وزارة التعليم
Ministry of Education

موهبة
Mawhiba

نسمو
NSMO

National Science and Mathematics Olympiad

Learning Materials for the Mathematic 02 Track

National Teams Competition 2026

Mathematic

بسم الله الرحمن الرحيم

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Introduction

Our outstanding sons and daughters,

We extend to you our warmest congratulations on reaching the **National Teams Stage**, which represents one of the most important milestones in your mathematical journey toward excellence at the national level.

In this resource packet, you will delve deeper into the four branches of mathematics: **Combinatorics, Geometry, Algebra, and Number Theory**, exploring topics such as **permutations and combinations, ratios and similarity, percentages and linear equations, and prime numbers and factors**.

This stage aims to empower you to apply mathematical concepts in constructing complex solutions, while developing skills in abstract thinking and precise problem analysis.

It is a stage in which you demonstrate teamwork and scientific collaboration, preparing you to engage in deeper levels of global mathematical competition.

Rise to the challenge, and continue your journey toward excellence with confidence and determination.

The Scientific Team for the National Science and Mathematics Olympiad (NSMO) – Mathematics Track

First Unit: ALGEBRA

Factoring

Factoring algebraic expressions means rewriting an expression as a product of simpler expressions, called factors.

Number of Terms in the Expression: **Binomial Expression**. (After factoring out the common factor, the expression becomes)	
Difference of two squares	$a^2 - b^2 = (a+b)(a-b)$
Difference of two cubes	$a^3 - b^3 = (a-b)(a^2+ab+b^2)$
Sum of two cubes	$a^3 + b^3 = (a+b)(a^2-ab+b^2)$

Number of Terms in the Expression: A Trinomial. ax^2+bx+c where a, b, c are constants and $a \neq 0$	
when $a = 1$	1) Before starting the factorisation process, the trinomial should be written in descending order according to the variable used. 2) We must look for a common factor among the algebraic terms and factor it out first, then proceed to factor the remaining expression. 3) If the sign of the last term (the constant term) is positive, then the signs of its factors will be the same and will follow the sign of the middle term in the original expression. $x^2+5x+6=(x+3)(x+2)$ $x^2-5x+6=(x-3)(x-2)$ 4) If the sign of the last term is negative, then the signs of its factors will be different, and we choose the pair whose difference matches the sign of the middle term. $x^2-5x-6=(x-6)(x+1)$ $x^2+5x-6=(x+6)(x-1)$
when $a \neq 0,1$	The expression is factored into two binomials as follows: 1) Factor the first term ax^2 into two factors whose product is ax^2. 2) Factor the last term c into two factors whose product is c. 3) (product of the outer terms + product of the inner terms) = the middle term bx.

Example 1:

Factor

$$2x^2 + 7x + 6$$

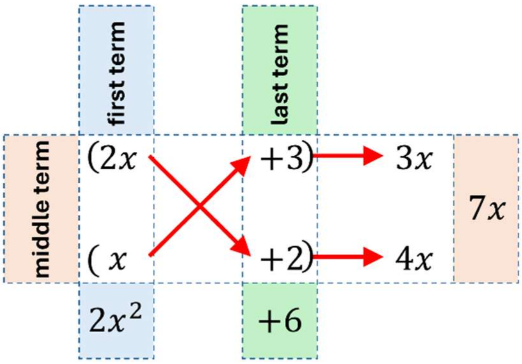
Solution:

first term $2x \cdot x = 2x^2$

last term $2 \cdot 3 = 6$

middle term $3x + 4x = 7x$

Therefore



$$2x^2 + 7x + 6 = (2x + 3)(x + 2)$$

Example 2:

Factor

$$2x^2 + x - 6$$

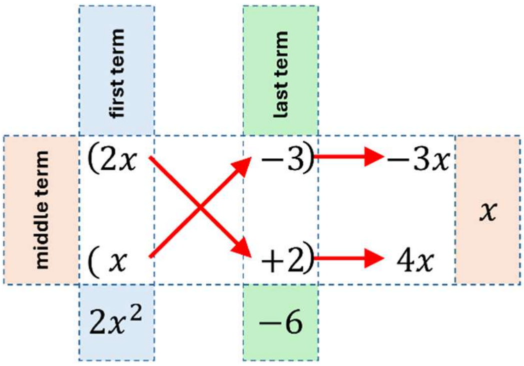
Solution:

first term $2x \cdot x = 2x^2$

last term $2 \cdot (-3) = -6$

middle term $-3x + 4x = x$

Therefore



$$2x^2 + 7x - 6 = (2x - 3)(x + 2)$$

Exercises:

Factor:

1) $x^2 - 8x + 7$	2) $x^2 - 6x - 7$
3) $x^2 - 25$	4) $2x^2 - 50$
5) $x^2 - 13x + 42$	6) $2x^2 + 5x + 2$
7) $x^3 - 1000$	8) $15x^3 + 7x^2 - 2x$
9) $5x^3 - 625$	10) $30x^4 + 5x^3 - 5x^2$
11) $x^2 - 2x + 1$	12) $x^2 + 10x + 25$
13) $6x^3y - 13x^2y + 6xy$	14) $2x^2 + 10x + 12$
15) $12x^2y^2 - 15xy^2 - 63y^2$	16) $24x^3 + 10x^2y - 50xy^2$
17) $9 - 4y^2$	18) $x^{12} - 1$
19) $\frac{1}{4}a^4 - \frac{1}{9}$	20) $y^6 - 81$

Squares of Binomials

If $a$and $b$are real numbers, then:

The square of a sum of two terms is given by the identity:

$$(a+b)^2 = a^2 + 2ab + b^2$$

The square of a difference of two terms is given by the identity:

$$(a-b)^2 = a^2 - 2ab + b^2$$

Example 1:

When Majed wants to square any number that ends with the digit 5, he uses the following steps:

1. Erase the 5 at the end of the number to obtain a number k .
2. Multiply k by $k+1$ and write the digits 25 at the end of the resulting product.

For example, we find that 65^2 by evaluating $6 \times 7 = 42$ and then placing 25 on the end of this product.

$$65^2 = 4225$$

How can that be correct?

Solution:

$$65^2 = (60+5)^2 = 60^2 + 10 \times 60 + 25 = 60(60+10) + 25$$
$$= 60 \times 70 + 25$$
$$= 4200 + 25 = 4225$$

And in the same way,

$$65^2 = (70-5)^2 = 70^2 - 10 \times 70 + 25 = 70(70-10) + 25$$
$$= 70 \times 60 + 25$$
$$= 4200 + 25 = 4225$$

Use this trick to compute 105^2 .

Exercises:

(1) In each of the following, complete the blank so that the expression becomes a perfect square:

$(a)\ x^2 - 6x + \cdots$
$(b)\ x^2 + 7x + \cdots$
$(c)\ x^2 - 0.4x + \cdots$
$(d)\ x^2 - \cdots + 42.25$

(2) Expand each of the following:

$(a)\ (y+5)^2$
$(b)\ (3z+8)^2$
$(c)\ (x-6)^2$
$(d)\ (-2y+9)^2$
$(e)\ (-x-9y)^2$
$(f)\ (2r-\frac{2}{r})^2$

(3) Find l if

$$5l^2 - 20l = 0$$

(4) Find l if

$$l^2 - 144 = 0$$

(5) Find w if

$$29 = (w-2)^2 - 7$$

(6) Find v if

$$94 - 5(v-3)^2 = 14$$

(7) Find e if

$$3(4+e)^2 - 40 = 68$$

(8) Find m if

$$m^2 - 6m + 9 = 0$$

(9) Find a if

$$a^2 + 36 = 12a$$

(10) Find t if

$$t^2 + 8t - 20 = 0$$

(11) Find h if

$$\frac{3(h-3)}{2}=\frac{27}{2h-6}$$

(12) Find x if

$$\frac{3x-6}{2}=\frac{27}{8x-16}$$

(13) Find y if

$$y^2+12y+32=0$$

(14) If $x-y=8$, $xy=-15$ Find

$a)\ x^2+y^2$ $\quad$ $b)\ (x+y)^2$ $\quad$ $c)\ x^4+y^4$

(15) If

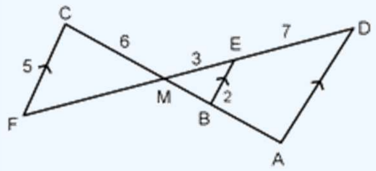
$x=2025^{1447}-2025^{-1447}$ and $y=2025^{1447}+2025^{-144}$

Find x^2-y^2 .

Second Unit: Geometry

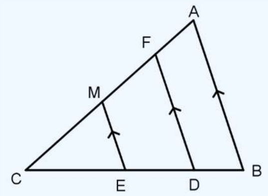
Revision Exercises

(1) In the adjacent figure, $AD \parallel BE \parallel FC$. Use the given lengths on the figure to find the lengths $\underline{MB}$,$\underline{MF}$, $\underline{AB}$ $\underline{AD}$



(2) In the adjacent figure, $AB \parallel FD \parallel ME$. Moreover, $AF: FM: MC = 2: 3: 5$. If we have that $ED = 7.5$ and $AF = 4$.

Find the length of $\underline{EC}$, $\underline{BD}$, $\underline{AC}$.

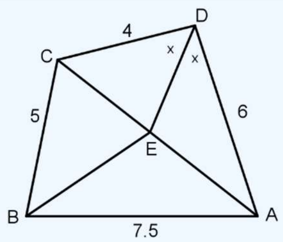


(3) Triangle ABC has side Lengths $\underline{AB}$, $\underline{BC}$, $\underline{CA}$ equal to 4,5,6 respectively. The angle bisector of $\angle A$ intersects $\underline{BC}$ at D. Find the lengths of $\underline{BD}$, $\underline{DC}$.

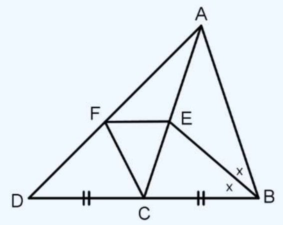
(4) Triangle ABC has side Lengths $\underline{AB}$, $\underline{BC}$, $\underline{CA}$ equal to 9,5,6 respectively. The external angle bisector of $\angle A$ intersects $BC^{\leftrightarrow}$ at D. Find the lengths of $\underline{BD}$, $\underline{DC}$.

(5) In triangle ABC, X is the midpoint of $\underline{BC}$. The angle bisector of $\angle AXB$ intersects $\underline{AB}$ at D. Similarly, The angle bisector of $\angle AXC$ intersects $\underline{AC}$ at E. Prove that, $\underline{DE} \parallel \underline{BC}$.

(6) In the adjacent figure, quadrilateral $ABCD$ has $\overrightarrow{DE}$ which bisects $\angle ADC$. Using the given lengths on the figure. Prove that $\overrightarrow{BE}$ bisects $\angle ABC$.



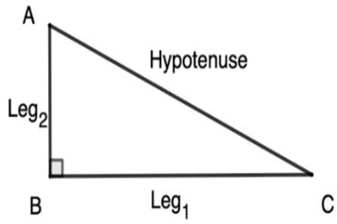
(7) In the adjacent figure, triangle ABC has $AB = AC$, D lies on the ray $\overrightarrow{BC}$ such that $BC = CD$. If $\overrightarrow{BE}$ bisects $\angle ABC$, and $EF \parallel BD$. Prove that $\overrightarrow{CF}$ bisects $\angle ACD$.



Pythagorean Theorem and Its Applications

A right-angled triangle is a triangle in which one of its angles equals 90° (a right angle). The side opposite the 90° angle is called the hypotenuse, and the other two sides are called the legs (the sides that form the right angle).

The Pythagorean theorem defines the relationship between these three sides in any right-angled triangle.



Theorem 1: (Pythagorean Theorem).

In a right-angle triangle, where a, b are the lengths of its legs and c is the length of its hypotenuse. The sum of the squares of the legs is equal to the square of the hypotenuse. That is:

$$\boxed{a^2 + b^2 = c^2}$$

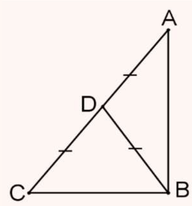
Theorem 2: (Pythagorean Theorem Inverse).

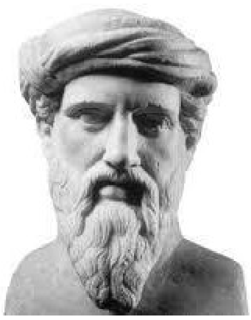
Any triangle with side lengths a, b, c that satisfy the relation $a^2 + b^2 = c^2$. Then, this triangle is a right angles triangle. Moreover, c is the length of the hypotenuse.

Theorem 3:

A triangle is right angled if one of the medians is equal to half of the side it is connected to. Moreover, that side will be the hypotenuse. For example:

.$(AD = BD = CD \Rightarrow \angle C = 90°)$





***Pythagoras** was a **Greek philosopher and mathematician** who lived approximately between **570 and 495 BC**. Born on the Greek island of **Samos**, he travelled extensively to many regions, including Greece, Egypt, and possibly India.

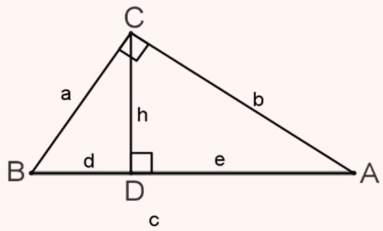
Around 530 BC, he settled in the Greek colony of **Croton, Italy**, where he established a school dedicated to the discussion of scientific and mathematical subjects. In his youth, he journeyed through **Mesopotamia** (modern-day Syria and Iraq) and spent time studying in **Egypt**. After two decades of travel and rigorous study, Pythagoras successfully absorbed all the mathematical knowledge known across the various major civilisations of the time.

Pythagoras is credited with proving his most famous theorem in geometry—which will be a subject of our study in this program—by relating the areas of the squares corresponding to the sides of a **right-angled triangle**. This fundamental theorem continues to be used by many engineers today in the process of land and building construction.

Theorem 4**:**

In the right angled triangle with sides a, b and hypotenuse c. If the altitude from the right angle to the hypotenuse $DC = h$ and $DA = e, DB = d$. Then:

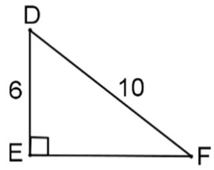
$$a^2 = dcb^2 = ech^2 = edhc = ab$$



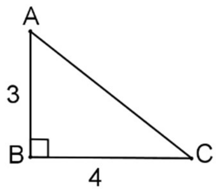
Examples:

Find the length of the unknown sides in the following triangles:

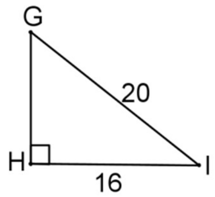
$$EF = \sqrt{DF^2 - DE^2} = \sqrt{100 - 36} = 8$$



$$AC = \sqrt{AB^2 + BC^2} = \sqrt{9 + 16} = 5$$



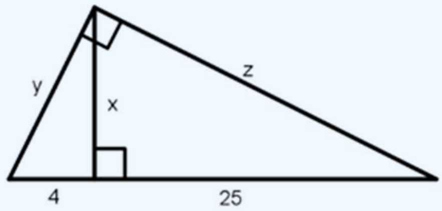
$$GH = \sqrt{GI^2 - HI^2} = \sqrt{144} = 12$$



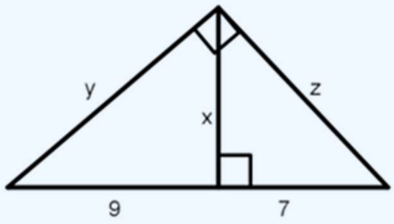
Exercises:

In $(1-7)$. Find the value of x, y, z.

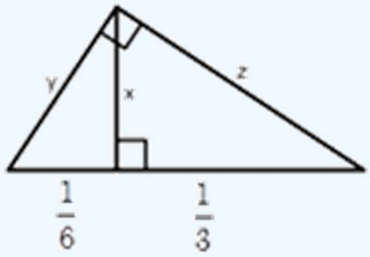
(1)



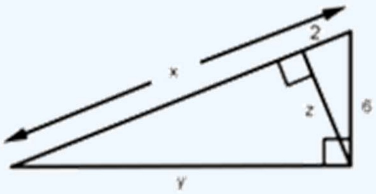
(2)



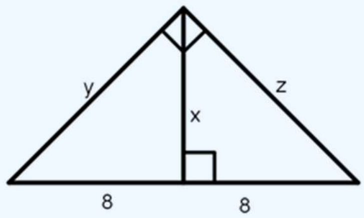
(3)



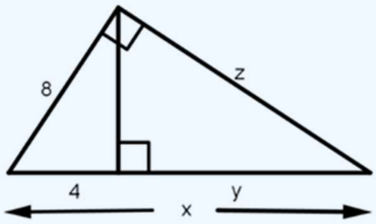
(4)



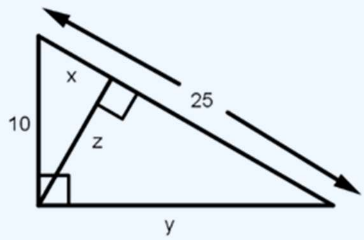
(5)



(6)



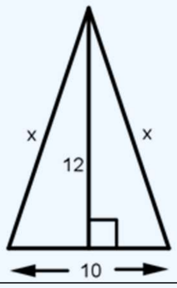
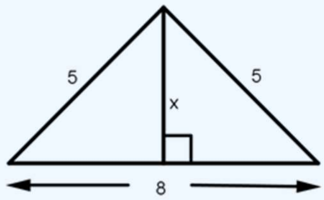
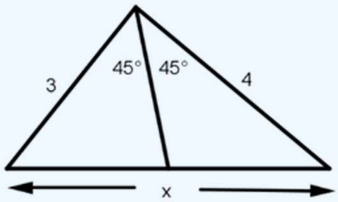
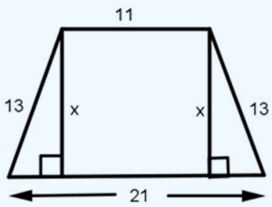
(7)



In $(8-11)$. You are given the length of a square's side. Find the length of its diagonal.

(8)	2	(9)	10	(10)	$20k$	(11)	$7n\sqrt{2}$

In $(12 - 15)$, find the value of x.

	(13)		(12)
	(15)		(14)

Special Triangles

Theorem 5:

In a right-angled triangle where the angles are equal to $30°, 60°, 90°$. The length of the side opposite the $30°$ angle is half the length of the hypotenuse. And the length of the side opposite to the $60°$ angle is $\frac{\sqrt{3}}{2}$ the length of the hypotenuse**.**

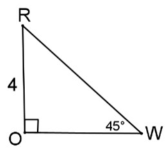
Theorem 6:

In the isosceles right-angled triangle, the length of the sides opposite to the $45°$ angle is $\frac{\sqrt{2}}{2}$ the length of the hypotenuse

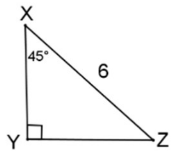
Example:

Find the length of the unknown side in the following triangles:

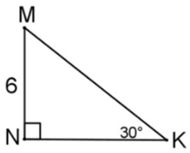
$$OW = 4, RW = \sqrt{16+16} = 4\sqrt{2}$$



$$YZ = YX = \frac{\sqrt{2}}{2} \cdot 6 = 3\sqrt{2}$$

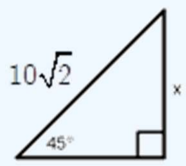
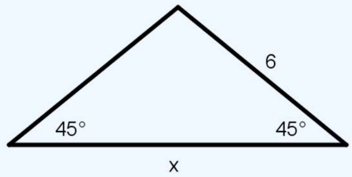
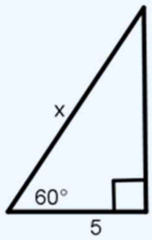
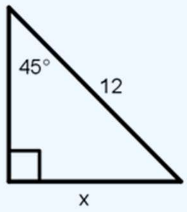
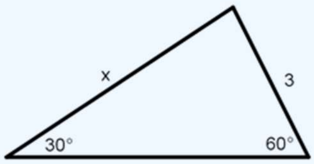
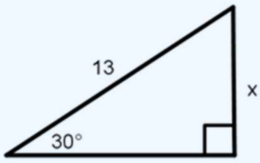
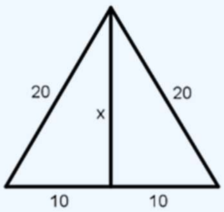


$$MK = 12, NK = \sqrt{144-36}$$
$$= 6\sqrt{3}$$

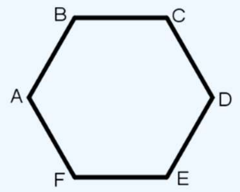


Exercises:

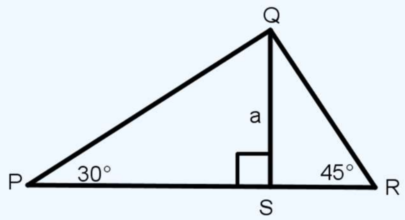
In exercises $(1-7)$, find the value of x.

	(2)		(1)
	(4)		(3)
	(6)		(5)
			(7)

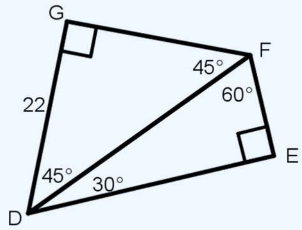
(8) In the adjacent figure: $ABCDEF$ is a regular hexagon with side length 8. Find the length of AC, AD



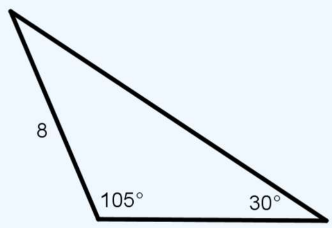
(9) In the following figure, find the lengths of PQ, PS, QR, in terms of a.



(10) In the adjacent figure, find the lengths of the unknown sides if possible.

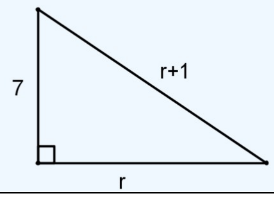


(11) In the adjacent figure, find the perimeter of the triangle.



Challenge Exercises:

(1) In the adjacent figure, find r.



(2) The triangle ABC has $\angle A = 90°$, $AB = AC$. If D lies on BC. Prove that $BD^2 + CD^2 = 2AD^2$.

(3) If the perimeter of a right-angled triangle is 30 cm and its area is 30 cm^2. Find the lengths of its sides.

(4) The triangle ABC has $\angle C = 90°$. AD is the internal angle bisector of $\angle A$ intersects BC at D. If

$$AB = 15\,, AC = 9\ \,, BD: DC = 5: 3$$

Find the distance from D to AB.

(5) In triangle ABC, we have $\angle C = 90°$, $BC = 6$, $AC = 12$. If the perpendicular bisector of AB intersects AB, BC at D, E respectively. Find the length of CE.

(6) In rectangle $ABCD$. We have $CE \perp DB$ at E. If $CE = 5$, $BE = \frac{1}{4}BD$, find the length of AC.

(7) In triangle ABC, we have $\angle C = 90°$. If point D is the midpoint of the segment AC. Prove that:

$$AB^2 + 3BC^2 = 4BD^2$$

(8) The triangle ABC has $\angle C = 90°$. If E, D lie on AC, BC respectively. Prove that

$$AD^2 + BE^2 = AB^2 + DE^2$$

Third Unit: Number Theory

Revision Exercises

1) Which of the following numbers are prime numbers: $73, 91, 101, 143, 199$?

2) Let p, q be different prime numbers. Find the number of different divisors for:

$$a)\ pq \qquad b)\ p^2q \qquad c)\ p^2q^2 \qquad d)\ p^nq^m$$

3) Prove that the multiplication of any three consecutive integers is divisible by 6.

4) Prove that the multiplication of any five consecutive integers is divisible by:

a) 30.

b) 120.

5) Find the least positive integer n such that 660 divides $n!$.

6) How many zeros are at the end of the number $10!$?

7) Let n be a natural number. Is it possible for the decimal representation of $(n!)$ to have exactly five consecutive zeros starting from the unit's digit??

8) Find all natural solutions x, y for the equation:

$$x^2 - y^2 = 33$$

The Greatest Common Divisor (gcd)

Definition:

Given a, b, c natural numbers such that $a \times b = c$. Then we say that a divides c and we denote that by $a|c$. Similarly, $b|c$. Moreover, we say that numbers a, b are divisors (or factors) of c. Finally, we also say that c is a multiple of a, b.

Definition:

The greatest common divisor (gcd for short) of a, b is the largest positive integer that divides both a and b

But how do we calculate the gcd of two numbers?

Example 1: Calculate $gcd\ (6,8)$.

Solution: We can look at the divisors of both 6 and 8:

$$d_6 = \{1,2,3,6\}$$
$$d_8 = \{1,2,4,8\}$$

Then, the common divisors for 6,8:

$$d_6 \cap d_8 = \{1,2\}$$

Thus, the greatest common divisor is 2.

We can see that this way uniquely determines the gcd. However, it may take a long time if the numbers were large. So we will look at an example with:

Another way to calculate the greatest common divisor:Example 2: Calculate $gcd\ (36,48)$.

Solution: we factorise the two numbers into their prime factorisations.

To get that:

36	2
18	2
9	3
3	3
	1

48	2
24	2
12	2
6	2
3	3
	1

$$36 = 2^2 \times 3^2 \ , \qquad 48 = 2^4 \times 3^1$$

Now, the way to obtain the gcd prime factorization is by **taking the common prime factors of both numbers with the smaller power from each factor.**

Apply this rule in this example to get:

$$gcd\ (36,48) = 2^2 . 3^1 = 12$$

Exercises:

1) Calculate $gcd\ (8,9)$.

2) Calculate $gcd\ (54,96)$.

3) Calculate $gcd\ (35,91)$.

4) Calculate $gcd\ (6,54)$.

5) Calculate $gcd\ (199,256)$.

6) The gcd of number n and 120 is 24. Which of the following could be the prime factorization of n?

$(a) 2 \times 3^3$

$(b) 2^2 \times 3^3$

$(c) 2^3 \times 3^2 \times 11$

$(d) 2^4 \times 3^3 \times 5$

The least common multiple (lcm)

Definition:

The least common multiple (lcm for short) of a, b is the smallest positive integer that is a multiple of both a and b.

Example 3: Calculate $lcm(6,8)$.

Solution: We calculate the multiples of both 6 and 8 to get:

$$m_6 = \{6,12,18,24,30,36,42,48,......\} m_8 = \{8,16,24,32,40,48,....\}$$

Then, we calculate the common multiples to obtain: $m_6 \cap m_8 = \{24,48,72,......\}$

Therefore, the least common multiple for 6,8 is 24.

Similar to the gcd, this way takes a long time, especially if the common multiple is far away (notice that the multiples are infinite). Thus, we look at:

Another way to calculate the least common multiple:

Example 4: Calculate $lcm(36,48)$.

Solution: we factorise the two numbers into their prime factorisations.

To get that:

36	2
18	2
9	3
3	3
	1

48	2
24	2
12	2
6	2
3	3
	1

$$36 = 2^2 \times 3^2 \ , \qquad 48 = 2^4 \times 3^1$$

Now, the way to obtain the lcm using the prime factorization is by **taking the common prime factors of both numbers with the higher power from each factor.** Apply this rule in this example to get:

$$lcm\ (36,48) = 2^4 . 3^2 = 144$$

Exercises:

1) Calculate $lcm\ (8,9)$.

2) Calculate $lcm\ (54,96)$.

3) Calculate $lcm\ (35,91)$.

4) Calculate $lcm\ (6,54)$.

5) Calculate $lcm\ (199,256)$.

6) The gcd of n and 20 is 180. Which of the following could be the prime factorisation of n?

$(a) 2 \times 3^3$

$(b) 2^2 \times\ 3^2$

$(c) 2^3 \times 3^2$

$(d) 2^4 \times 3^3 \times 5$

Important rule:

Given two natural numbers a, b, with $d = gcd\ (a, b)$ and $m = lcm(a, b)$. Then,

$$m. d = a. b$$

Example 1:

If the $gcd\ (n, 8) = 2$, and $lcm(8, n) = 24$. Then what is the value of n?

Solution:

Using the rule, we get that:

$$n \times 8 = 2 \times 24 \ \Rightarrow n = 6.$$

Perfect Square:

Is a positive integer such that all the prime factors in its prime factorization have **even** power.

For example:

The number A that has the prime factorisation:

$$A = p_1^{\alpha_1}. p_2^{\alpha_2} .... p_n^{\alpha_n}$$

Is a perfect square if and only if $\alpha_1, \alpha_2, ..., \alpha_n$ were **even** numbers.

Example 2:

Which of the following numbers are perfect squares:

$$144,\ 128,\ 1024,\ 360$$

Solution:

By taking the prime factorization we find that 144, 1024 are perfect squares while 128,360 are not since they have primes with odd powers (show it on your own!).

Exercises:

1) If the $gcd\ (n, 18) = 6$ and $lcm(18, n) = 36$. Then what is the value of n?

2) Which of the following numbers are perfect squares:

$196,\ 192,\ 169,\ 240$

3) Challenge: Can a number that is composed of 100 zeros, 100 ones, 100 twos be a perfect square?

4) Challenge: Prove that the number with odd number of divisors must be a perfect square.

5) Challenge: Prove that for any natural numbers a, b, with $d = gcd\ (a, b)$ and $m = lcm(a, b)$. Then,

$$m.d = a.b$$

Fourth Unit: Combinatorics

Properties of Permutations

Permutations with Repetition:

(1) Find the number of permutations of the letters of the word $PARALLEL$.

In the previous exercise, we noticed that some arrangements are counted repeatedly when using the ordinary permutation method, because swapping the first and second A makes no real difference, and similarly for the letter L.

Permutations with repetition:

If there are n objects where one item is repeated r_1 times, a second item r_2 times, and a third r_3 times, then the number of permutations is:

$$\frac{n!}{r_1! \times r_2! \times r_3!}$$

Exercises:

(2) How many ways are there to rearrange the letters of $abcaadbddd$?

(3) How many ways are there to rearrange the digits of 456733727?

(4) We have the digits *1, 2, 3, 1, 4, 5, 5*:

(a) How many numbers can be formed by arranging all these digits?

(b) How many numbers can be formed by arranging all these digits, provided the number begins and ends with *5*?

(5) How many words can be formed from the letters of the word $SAUD$?

(6) In how many different ways can the letters of $SALMAN$ be arranged?

(7) How many different ways can the letters of the word $ARABIA$ be arranged if the letters B must be between two A's (not necessarily next to them)?

(8) How many different words can be formed from the letters of $ELEMENTARY$ if the three E's appear together?

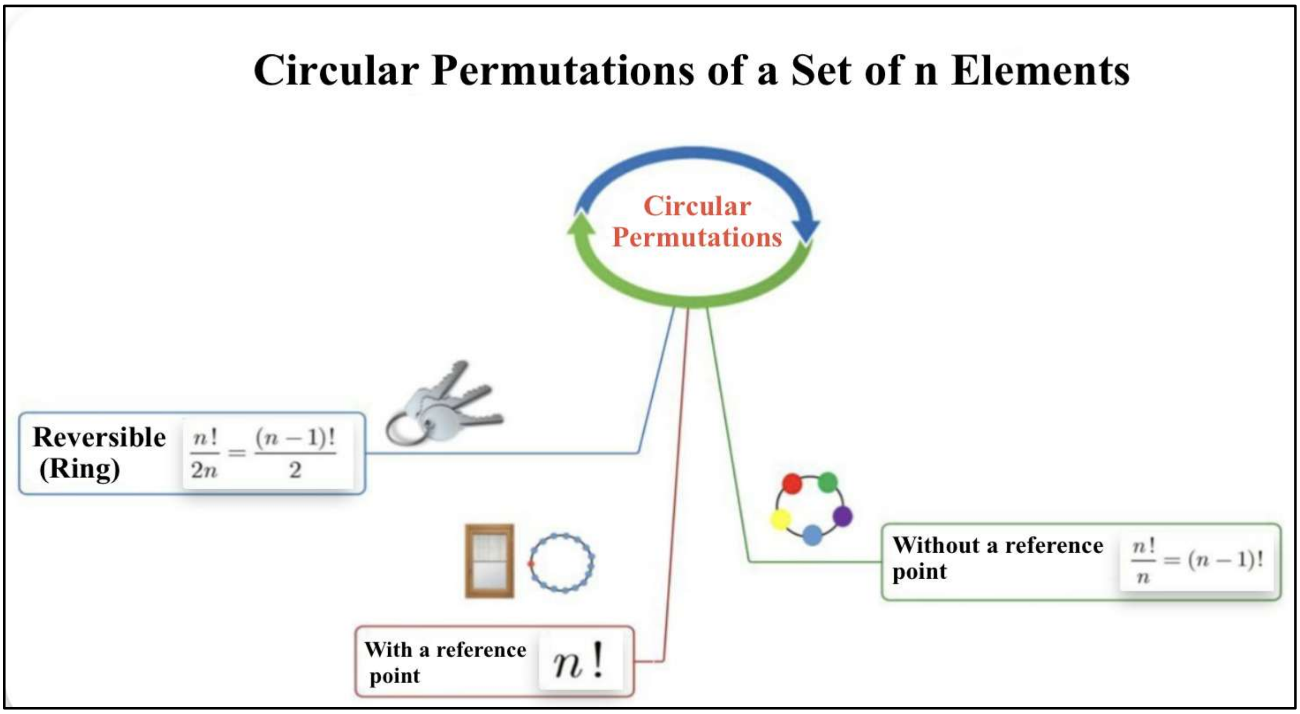
(9) How many ordered triples of positive integers (a, b, c) satisfy: $a \times b \times c = 231$

Circular Permutations

(10) In how many ways can we arrange five spice jars (salt, ginger, cumin, pepper, thyme) in a circle?

Circular permutation rule: The number of distinct arrangements of n objects placed around a circle is:

$$\frac{n!}{n} = (n-1)!$$



Exercises:

(11) In how many ways can we arrange 5 objects around a circle? 7 objects around a circle? n objects around a circle?

(12) In how many ways can eight people sit around a round table if one chair is red and the rest are black?

(13) In how many ways can we arrange:

(a) 36 people around a circle?

(b) 7 players in a circle if one of them is standing behind the referee?

(14) In how many ways can 5 students sit around a round table? And how many ways are there if one of the chairs is under the window?

(15) A circular necklace is made up of different types of beads. How many different necklaces can be formed from 13 beads:

(a) If flipping (reflection) is not allowed?

(b) If flipping is allowed?

(16) In how many ways can 5 engineers and 5 doctors be seated around a round table so that no two engineers sit next to each other?

(17) In how many ways can 7 engineers and 5 doctors be seated around a round table so that no two doctors sit next to each other? And how many ways if no two engineers are allowed to sit next to each other?

Combinations

Combinations: The number of ways to choose r items from a set of n items without regard to order and without repetition is $\frac{n!}{(n-r)!\times r!}$

and this number is denoted by $_nC_r$ or $\binom{n}{r}$

(*18*) Five members of the Saudi team want to choose *2* of them at random to form a rules committee. In how many ways can they do that?

Properties of combinations and permutations

Compute the following and write your observations:

(a $(6)!$ and $2!+4!$ (b $_7P_3$ and $\frac{7!}{4!}$ (c $_8P_3$ and $3!\cdot\binom{8}{3}$ (d $\binom{8}{3}$ and $\binom{8}{5}$

Note: From part (a) of the previous example we find *that* $(2+4)! \neq 2!+4!$

So we conclude that the statement $(n+m)! \neq n!+m!$ is not always true.

And from parts (b), (c), and (d) we observe the equalities:

$$_7P_3 = \frac{7!}{4!} \text{ ، and } \quad _8P_3 = 3!\cdot\binom{8}{3} \text{ and } \quad \binom{8}{5} = \binom{8}{8-5} = \binom{8}{3}$$

We will find later that these relations are always true for any positive integers m, n such that $m \leq n$

$$n! = n\cdot(n-1)!$$

$$_nP_m = m!\cdot\binom{n}{m}$$

$$\binom{n}{m} = \binom{n}{n-m}$$

Exercises:

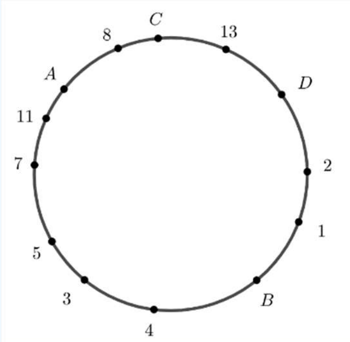
(19) How many different words can be formed from rearranging the letters of $ELEMENTARY$ such that no two E's appear together?

(20) How many different 10-letter words can be formed using only the letters E and F, if the number of E's is greater than the number of F's?

(21) Points are labeled on the circumference of a circle as shown. We will draw chords according to these conditions:

1. Draw a chord between any two points whose names are letters.
2. Draw a chord between any two points whose names are odd numbers.
3. Draw a chord between any two points whose names are even numbers.

How many chords can be drawn in total?



(22) In how many ways can 8 students be divided into:

(a) Two distinct groups of 3 students each, with 2 students left unassigned?

(Note: In the next four cases, the groups are not distinct.)

(b) Two groups of 4 students each?

(c) Four groups of 2 students each?

(d) One group of 4 students and two groups of 2 students?

(e) One group of 4 students, one group of 3 students, and one student left unassigned?

(23) Osama wants to sit with 10 of his friends around a round table. How many different pairs of Osama's friends can be seated next to him?

Solutions

Algebra Solutions

Factoring:

Exercises:

(1)

$$x^2 - 8x + 7 = (x-1)(x-7)$$

(2)

$$x^2 - 6x - 7 = (x-7)(x+1)$$

(3)

$$x^2 - 25 = (x-5)(x+5)$$

(4)

$$2x^2 - 50 = 2(x^2 - 25) = 2(x-5)(x+5)$$

(5)

$$x^2 - 13x + 42 = (x-7)(x-6)$$

(6)

$$2x^2 + 5x + 2 = (2x+1)(x+2)$$

(7)

$$x^3 - 1000 = x^3 - 10^3 = (x-10)(x^2 + 10x + 100)$$

(8)

$$15x^3 + 7x^2 - 2x = x(15x^2 + 7x - 2) = x(5x-1)(3x+2)$$

(9)

$$5x^3 - 625 = 5(x^3 - 125) = 5(x^3 - 5^3) = 5(x-5)(x^2 + 5x + 25)$$

(10)

$$30x^4 + 5x^3 - 5x^2 = 5x^2(6x^2 + x - 1) = 5x^2(2x+1)(3x-1)$$

(11)

$$x^2 - 2x + 1 = (x-1)(x-1) = (x-1)^2$$

(12)

$$x^2 + 10x + 25 = (x+5)(x+5) = (x+5)^2$$

(13)

$$6x^3y - 13x^2y + 6xy = xy(6x^2 - 13x + 6) = xy(3x-2)(2x-3)$$

(14)

$$2x^2 + 10x + 12 = 2(x^2 + 5x + 6) = 2(x + 3)(x + 2)$$

(15)

$$12x^2y^2 - 15xy^2 - 63y^2 = 3y^2(4x^2 - 5x - 21) = 3y^2(x - 3)(4x + 7)$$

(16)

$$24x^3 + 10x^2y - 50xy^2 = 2x(12x^2 + 5xy - 25y^2)$$
$$= 2x(4x - 5y)(3x + 5y)$$

(17)

$$9 - 4y^2 = 3^2 - (2y)^2 = (3 - 2y)(3 + 2y)$$

(18)

$$x^{12} - 1 = (x^6 - 1)(x^6 + 1) = (x^3 - 1)(x^3 + 1)((x^2)^3 + 1)$$
$$= (x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1)(x^2 + 1)(x^4 - x^2 + 1)$$

(19)

$$\frac{1}{4}a^4 - \frac{1}{9} = \left(\frac{a^2}{2}\right)^2 - \left(\frac{1}{3}\right)^2 = \left(\frac{a^2}{2} - \frac{1}{3}\right)\left(\frac{a^2}{2} + \frac{1}{3}\right) = \frac{1}{36}(3a^2 - 2)(3a^2 + 2)$$

(20)

$$y^6 - 81 = (y^3)^2 - 9^2 = (y^3 - 9)(y^3 + 9)$$

Squares of Binomials:
Exercises:

(1)

By using the identity:

$$x^2 \pm bx + \left(\frac{b}{2}\right)^2 = \left(x \pm \frac{b}{2}\right)^2$$

We get

$$(a)\ x^2 - 6x + 9$$

$$(b)\ x^2 + 7x + \frac{49}{4}$$

$$(c)\ x^2 - 0.4x + 0.04$$

$$(d)\ x^2 - 13x + 42.25$$

(2)

$$(a)\ (y+5)^2 = y^2 + 10y + 25$$

$$(b)\ (3z+8)^2 = 9z^2 + 48z + 64$$

$$(c)\ (x-6)^2 = x^2 - 12x + 36$$

$$(d)\ (-2y+9)^2 = 4y^2 - 36y + 81$$

$$(e)\ (-x-9y)^2 = x^2 + 18xy + 81y^2$$

$$(f)\ (2r - \frac{2}{r})^2 = 4r^2 - 8 + \frac{4}{r^2}$$

(3)

$$5l^2 - 20l = 0$$

$$\Longrightarrow 5l(l-4) = 0$$

$$\Longrightarrow 5l = 0 \quad or \quad l - 4 = 0$$

$$\Longrightarrow l = 0 \quad or \quad l = 4$$

(4)

$$l^2 - 144 = 0$$

$$\Rightarrow l^2 - 12^2 = 0$$

$$\Rightarrow (l - 12)(l + 12) = 0$$

$$\Rightarrow l - 12 = 0 \quad or \quad l + 12 = 0$$

$$\Rightarrow l = \pm 12$$

(5)

$$29 = (w - 2)^2 - 7$$

$$\Rightarrow (w - 2)^2 = 36$$

$$\Rightarrow w - 2 = 6 \qquad or \qquad w - 2 = -6$$

$$\Rightarrow w = 8 \qquad or \qquad w = -6$$

(6)

$$94 - 5(v - 3)^2 = 14$$

$$\Rightarrow 5(v - 3)^2 = 80$$

$$\Rightarrow (v - 3)^2 = 16$$

$$\Rightarrow v - 3 = 4 \qquad or \qquad v - 3 = -4$$

$$\Rightarrow v = 7 \qquad or \qquad v = -1$$

(7)

$$3(4 + e)^2 - 40 = 68$$

$$\Rightarrow 3(4 + e)^2 = 108$$

$$\Rightarrow (4 + e)^2 = 36$$

$$\Rightarrow 4 + e = 6 \qquad or \qquad 4 + e = -6$$

$$\Rightarrow e = 2 \qquad or \qquad e = -10$$

(8)

$$m^2 - 6m + 9 = 0$$

$$\Longrightarrow (m-3)^2 = 0$$

$$\Longrightarrow m - 3 = 0$$

$$\Longrightarrow m = 3$$

(9)

$$a^2 + 36 = 12a$$

$$\Longrightarrow a^2 - 12a + 36 = 0$$

$$\Longrightarrow (m-6)^2 = 0$$

$$\Longrightarrow m - 6 = 0$$

$$\Longrightarrow m = 6$$

(10)

$$t^2 + 8t - 20 = 0$$

$$\Longrightarrow (t-2)(t+10) = 0$$

$$\Longrightarrow t - 2 = 0 \qquad or \qquad t + 10 = 0$$

$$\Longrightarrow t = 2 \qquad or \qquad t = -10$$

(11)

$$\frac{3(h-3)}{2} = \frac{27}{2h-6}$$

$$\Longrightarrow (h-3)(h-3) = \frac{2 \cdot 27}{2 \cdot 3}$$

$$\Longrightarrow (h-3)^2 = 9$$

$$\Longrightarrow h - 3 = 3 \qquad or \qquad h - 3 = -3$$

$$\Longrightarrow h = 6 \qquad or \qquad h = 0$$

(12)

$$\frac{3x-6}{2}=\frac{27}{8x-16}$$

$$\Rightarrow (x-2)(x-2)=\frac{2\cdot 27}{8\cdot 3}$$

$$\Rightarrow (x-2)^2=\frac{9}{4}$$

$$\Rightarrow x-2=\frac{3}{2} \qquad or \qquad x-2=-\frac{3}{2}$$

$$\Rightarrow x=\frac{7}{2} \qquad or \qquad x=\frac{1}{2}$$

(13)

$$y^2+12y+32=0$$

$$\Rightarrow (y+8)(y+4)=0$$

$$\Rightarrow y+8=0 \qquad or \qquad y+4=0$$

$$\Rightarrow y=-8 \qquad or \qquad y=-4$$

(14)

If $x - y = 8$ and $xy = -15$ Find:

$a)\ x^2 + y^2$ $b)\ (x+y)^2$ $c)\ x^4 + y^4$

$$a)\ (x-y)^2 = 8^2$$

$$\Longrightarrow x^2 - 2xy + y^2 = 64$$

$$\Longrightarrow x^2 - 2(-15) + y^2 = 64$$

$$\Longrightarrow x^2 + y^2 = 64 - 30 = 34$$

$$b)\ (x+y)^2 = x^2 + 2xy + y^2$$

$$\Longrightarrow\ (x+y)^2 = x^2 + 2(-15) + y^2$$

$$\Longrightarrow\ (x+y)^2 = x^2 + y^2 - 30$$

$$\Longrightarrow\ (x+y)^2 = 34 - 30 = 4$$

$$c)\ (x^2+y^2)^2 = 34^2$$

$$\Longrightarrow x^4 + 2(xy)^2 + y^4 = 34^2$$

$$\Longrightarrow x^4 + y^4 = 1156 - 450 = 706$$

(15)

$$x - y = 2025^{1447} - 2025^{-1447} - 2025^{1447} - 2025^{-1447} = -\frac{2}{2025^{1447}}$$

$$x + y = 2025^{1447} - 2025^{-1447} + 2025^{1447} + 2025^{-1447} = 2 \cdot 2025^{1447}$$

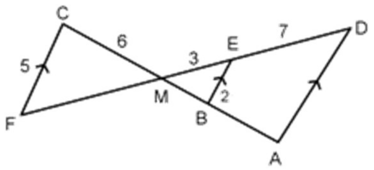
$$\because x^2 - y^2 = (x-y)(x+y)$$

$$\therefore x^2 - y^2 = \left(-\frac{2}{2025^{1447}}\right)(2 \cdot 2025^{1447}) = -4$$

Geometry Solutions

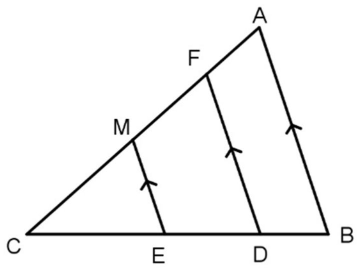
Revision Exercises Solutions

(1)



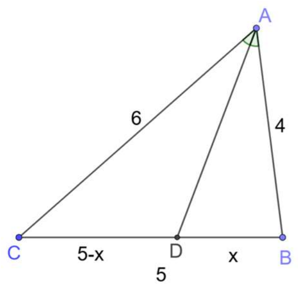
$$\frac{ME}{MF}=\frac{EB}{FC}\Longrightarrow MF=\frac{5\times3}{2}=7.5\,\frac{MB}{MC}=\frac{EB}{FC}\Longrightarrow MB=\frac{6\times2}{5}=2.4\,\frac{ME}{ED}=\frac{MB}{BA}\Longrightarrow BA=\frac{7\times2.4}{3}=5.6\,\frac{BE}{AD}=\frac{ME}{MD}\Longrightarrow AD=\frac{2\times10}{3}=\frac{20}{3}$$

(2)



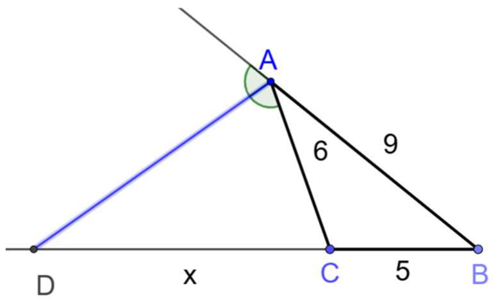
$$\frac{MC}{EC}=\frac{MF}{ED}\Longrightarrow CE=\frac{5\times7.5}{3}=12.5\,\frac{AF}{DB}=\frac{MF}{ED}\Longrightarrow DB=\frac{2\times7.5}{3}=5\,\frac{AF}{AC}=\frac{2}{10}\Longrightarrow AC=\frac{4\times10}{2}=20$$

(3)



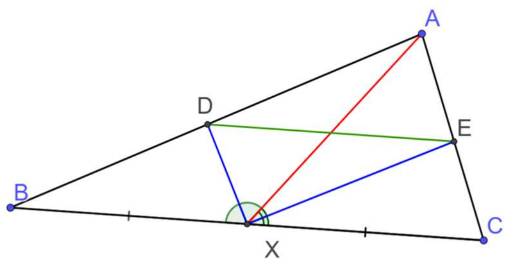
$$\frac{4}{6}=\frac{BD}{DC}\Longrightarrow\frac{4}{6}=\frac{x}{5-x}$$
$$\Longrightarrow BD=x=2$$

(4)



$$\frac{9}{6}=\frac{5+x}{5}\Longrightarrow DC=2.5\,,BD=7.5$$

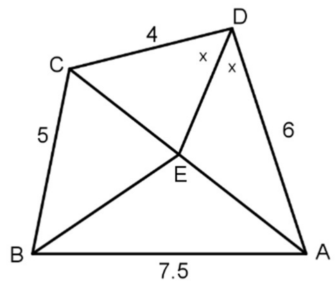
(5)



$$\frac{AX}{XC} = \frac{AE}{EC} \quad (1) \quad \frac{AX}{XB} = \frac{AD}{DB} \quad (2)$$

$$XB = XC \Longrightarrow \frac{AE}{EC} = \frac{AD}{DB} \Longrightarrow ED \parallel BC$$

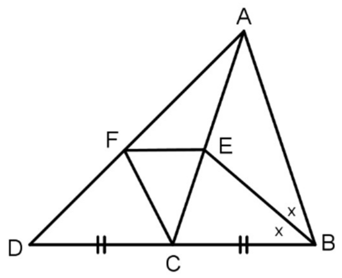
(6)



$$\frac{AE}{EC} = \frac{AD}{DC} = \frac{6}{4} = \frac{3}{2} \quad (1) \frac{AB}{BC} = \frac{7.5}{5} = \frac{3}{2} \quad (2)(1),(2) \Longrightarrow \frac{AB}{BC} = \frac{AE}{EC}$$

From the inverse of the angle bisector theorem we have BE is the angle bisector of $\angle ABC$.

(7)



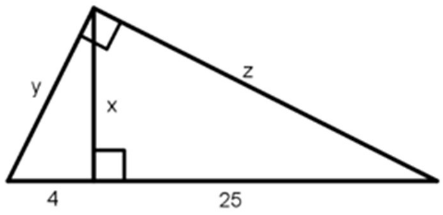
We have $AB = AC$, $BC = DC$. Using angle bisector theorem for $\angle ABC$, we get that:

$$\frac{AE}{EC} = \frac{AB}{BC} = \frac{AC}{DC} \quad (1) \frac{AE}{EC} = \frac{AF}{FD} \quad (2)(1),(2) \Longrightarrow \frac{AF}{FD} = \frac{AC}{DC}$$

From the inverse of the angle bisector theorem we have CF is the angle bisector of $\angle ACD$.

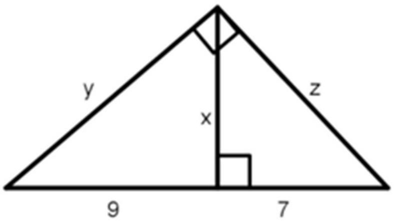
Exercise Solutions for Pythagorean Theorem:

(1)



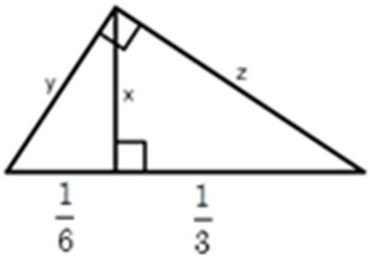
$$x^2 = 4 \times 25 \Longrightarrow x = 10$$
$$y^2 = 4 \times 29 \Longrightarrow y = 2\sqrt{29}$$
$$z^2 = 29 \times 25 \Longrightarrow z = 5\sqrt{29}$$

(2)



$$x^2 = 7 \times 9 \Longrightarrow x = 3\sqrt{7}$$
$$y^2 = 9 \times 16 \Longrightarrow y = 12$$
$$z^2 = 7 \times 16 \Longrightarrow z = 4\sqrt{7}$$

(3)



$$x^2 = \frac{1}{3} \times \frac{1}{6} \Longrightarrow x = \frac{\sqrt{2}}{6}$$

$$y^2 = \frac{1}{6} \times \frac{1}{2} \Longrightarrow y = \frac{\sqrt{3}}{6}$$

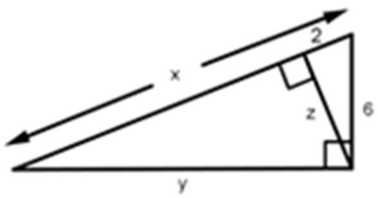
$$z^2 = \frac{1}{3} \times \frac{1}{2} \Longrightarrow z = \frac{\sqrt{6}}{6}$$

(4)

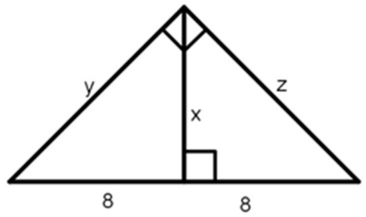
$$6^2 = 2 \times x \Longrightarrow x = 18$$

$$y^2 = 18 \times 16 \Longrightarrow y = 12\sqrt{2}$$

$$z^2 = 2 \times 16 \Longrightarrow z = 4\sqrt{2}$$



(5)

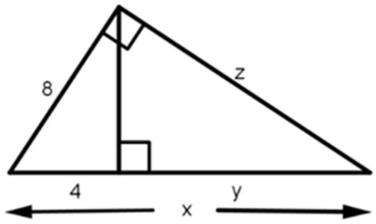


$$x^2 = 8 \times 8 \Longrightarrow x = 8$$

$$y^2 = 8 \times 16 \Longrightarrow y = 8\sqrt{2}$$

$$z = y = 8\sqrt{2}$$

(6)

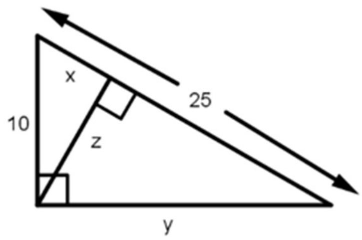


$$8^2 = 4 \times x \Longrightarrow x = 16$$

$$y = 16 - 4 \Longrightarrow y = 12$$

$$z^2 = 12 \times 16 \Longrightarrow z = 8\sqrt{3}$$

(7)



$$10^2 = x \times 25 \Longrightarrow x = 4$$

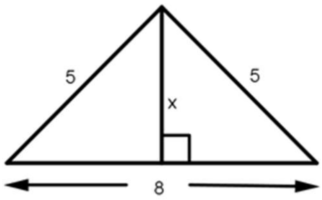
$$y^2 = 21 \times 25 \Longrightarrow y = 5\sqrt{21}$$

$$z^2 = 4 \times 21 \Longrightarrow z = 2\sqrt{21}$$

(8 – 11)

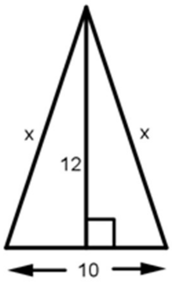
Square diagonal	Square side	Exercise
$2\sqrt{2}$	2	**(8)**
$10\sqrt{2}$	10	**(9)**
$20k\sqrt{2}$	$20k$	**(10)**
$14n$	$7n\sqrt{2}$	**(11)**

(12)



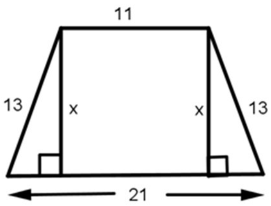
$$x = \sqrt{5^2 - 4^2} = 3$$

(13)



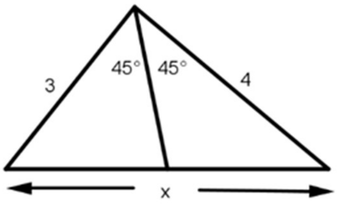
$$x = \sqrt{12^2 + 5^2} = 13$$

(14)



$$x = \sqrt{13^2 - 5^2} = 12$$

(15)



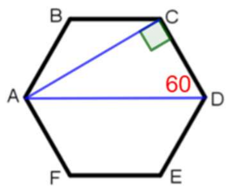
$$x = \sqrt{4^2 + 3^2} = 5$$

Exercise solutions for Special Triangles:

$(1 - 7)$

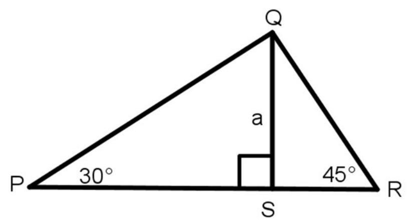
Value of x	Exercise Number
$6\sqrt{2}$	1
10	2
$6\sqrt{2}$	3
10	4
6.5	5
$3\sqrt{3}$	6
$10\sqrt{3}$	7

(8)



$$AC = 8\sqrt{3}$$
$$AD = 16$$

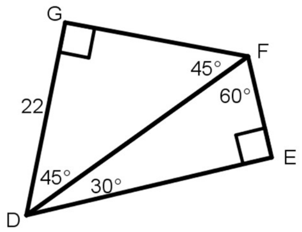
(9)



$$QR=a2$$
$$PS=a3$$
$$PQ=2a$$

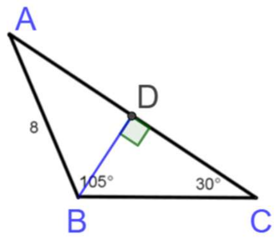
(10)

$$GF = 22DF = 22\sqrt{2}FE = 11\sqrt{2}ED = 11\sqrt{6}$$



(11)

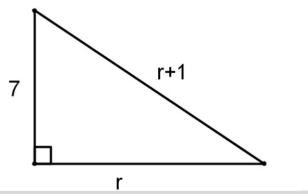
$$AB = 8 \quad , \quad AD = 4\sqrt{2} \quad , \quad BD = 4\sqrt{2}BC$$
$$= 8\sqrt{2} \quad , \quad DC = 4\sqrt{6} \Longrightarrow Perimeter$$
$$= 8 + 12\sqrt{2} + 4\sqrt{6}$$



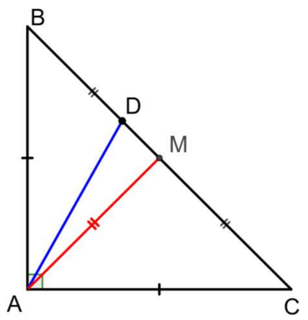
Challenge Exercises Solutions:

(1)

$$(r+1)^2 = r^2 + 7^2 \Longrightarrow r = 24$$



(2)



We draw an altitude from A on the hypotenuse BC. From the properties of isosceles triangles, the altitude will bisect BC at M. Moreover, from the properties of the median in the right angle, we have:

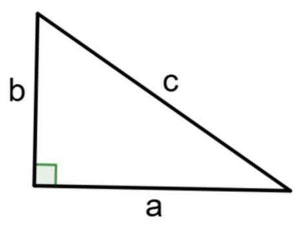
$$AM = CM = BM$$

$$BD^2 + CD^2 = (BM - MD)^2 + (CM + MD)^2$$
$$= BM^2 + MD^2 - 2BM \cdot MD + CM^2 + MD^2$$
$$+ 2CM \cdot MD = 2(AM^2 + MD^2) = 2AD^2$$

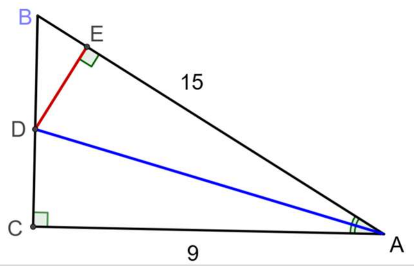
(3)

$$a^2 + b^2 = c^2$$

$$\frac{1}{2}a \cdot b = 30 \Longrightarrow a \cdot b = 60a + b + c = 30 \Longrightarrow a + b = 30 - c$$
$$\Longrightarrow (a+b)^2 = (30-c)^2 \Longrightarrow a^2 + b^2 + 2a \cdot b$$
$$= 900 + c^2 - 60c \Longrightarrow c^2 + 2 \cdot 60 = 900 + c^2 - 60c$$
$$\Longrightarrow 120 = 900 - 60c \Longrightarrow c = 13a \cdot b = 60 \ \Longrightarrow b$$
$$= \frac{60}{a}a^2 + b^2 = c^2 \Longrightarrow a^2 + \left(\frac{60}{a}\right)^2 = 13^2$$
$$\Longrightarrow a^4 - 169a^2 + 3600 = 0$$
$$\Longrightarrow (a^2 - 25)(a^2 - 144) = 0 \Longrightarrow a^2 = 25$$
$$\Longrightarrow a = 5\,, b = 12 or\ a^2 = 144 \Longrightarrow a = 12\ \,, b$$
$$= 5$$

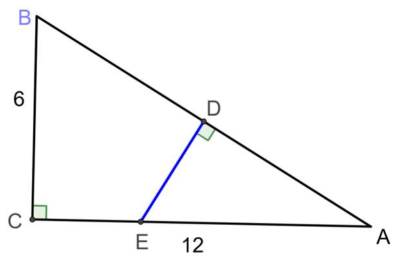


(4)



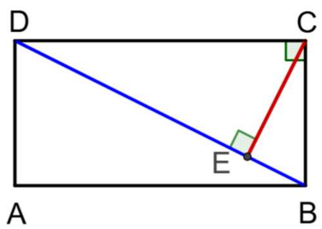
$$BC = \sqrt{225-81} = 12$$
$$\frac{AB}{AC} = \frac{BD}{DC} = \frac{3}{5} = \frac{BD}{12-BD} \Longrightarrow BD = 4.5\ , \qquad CD = 7.5$$
$$\Delta ADE \cong \Delta ADC\ \ (ASA)$$
$$\Longrightarrow DE = DC = 4.5$$

(5)



$$AB = \sqrt{144+36} = 6\sqrt{5} \Longrightarrow AD = DB = 3\sqrt{5}$$
$$\Delta ADE \sim \Delta ACB\ \ (AA)$$
$$\Longrightarrow \frac{AD}{AC} = \frac{AE}{AB} \Longrightarrow \frac{3\sqrt{5}}{12} = \frac{AE}{6\sqrt{5}} \Longrightarrow AE = 7.5$$
$$\Longrightarrow CE = 12 - 7.5 \Longrightarrow CE = 4.5$$

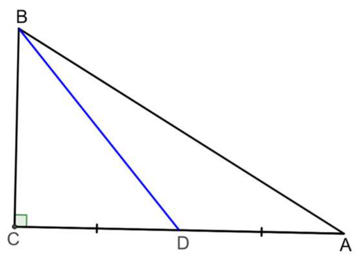
(6)



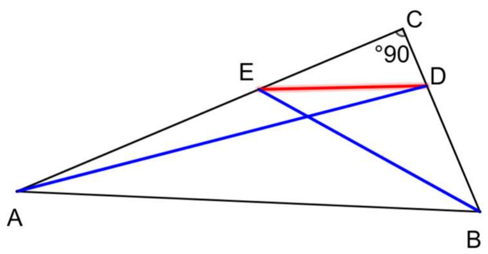
$$CE^2 = BE \cdot DE \Longrightarrow 25 = \frac{1}{4}BD \cdot \frac{3}{4}BD$$
$$\Longrightarrow BD = \frac{20\sqrt{3}}{3}$$

(7)

$$AB = 2CD$$
$$AB^2 + 3BC^2 = BC^2 + (2CD)^2 + 3BC^2$$
$$= 4(BC^2 + CD^2) = 4BD^2$$



(8)



$$(1) \quad \{BC^2 + CE^2 = BD^2 \; AC^2 + CD^2 = AD^2$$
$$(2) \quad \{CD^2 + CE^2 = ED^2 \; AC^2 + BC^2 = AB^2$$

By comparing the sum of (1) and the sum (2) We find that:

$$AD^2 + BE^2 = AB^2 + DE^2$$

Solutions for Revision Exercises:

(1)

- 73 is a prime number.
- 91 is a composite number since $91 = 7 \times 13$.
- 101 is a prime number.
- 143 is a composite number since $143 = 1 \times 13$.
- 199 is a prime number.

(2)

From the law of the number of divisors we get:

a) $(1+1)(1+1) = 4$

b) $(2+1)(1+1) = 6$

c) $(2+1)(2+1) = 9$

d) $(m+1)(n+1) = mn + m + n + 1$

(3)

Notice that in any three consecutive integers. We have one even and one that is divisible by 3 (they could be the same number). Thus, the multiplication must be divided by 6.

(4)

(a) We factorize

$$30 = 2 \times 3 \times 5.$$

We need to prove that the product is divisible by **2, 3, and 5** simultaneously.

- **Divisibility by 5:** In any sequence of five consecutive natural numbers, exactly one number must be divisible by 5.
- **Divisibility by 3:** In any sequence of three consecutive natural numbers, exactly one number must be divisible by 3. Since we have five numbers, this condition is satisfied.
- **Divisibility by 2:** In any sequence of two consecutive natural numbers, exactly one number must be divisible by 2. Since we have five numbers, this condition is also satisfied.

Since the product is divisible by **2, 3, and 5**, it is divisible by

$$2 \times 3 \times 5 = 30.$$

(b) Notice that it is enough to show that the multiplication is divisible by 120 (since 30 divides 120). The factorization of 120 is:

$$120 = 2^3 \times 3 \times 5$$

Thus, we need to show that the number is divisible by $3,5,8$. Similar to the previous question, notice that any five consecutive integers must have a multiple of 3 and a multiple of 5. Therefore, we only need to show that the multiplication is also a multiple of 8. Notice that in any 4 consecutive numbers, we have two even integers. Moreover, one of them must be a multiple of 4. Thus the multiplication must be divisible by 120.

.

(5)

By factorizing 660, we get:

660=4×3×5×11.

Thus, we need to make sure that $n!$ contains the primes 2,3,5,11. This means that n≥11. By substituting $n = 11$ it is clear that 660 divides $11!$

(6)

The number of consecutive zeros at the end of any number depends on how many times does 10 divide the number. However, notice that $10 = 2 \times 5$. Thus, we need to look for the number of $5's$ and $2's$ in $10!$. We can clearly see that 5 only appears twice at 5,10, while 2 appears more than twice. This means there is exactly two $10's$ that divide $10!$.
This gives an answer of 2.

(7)

Not possible. Notice that $24!$ ends with exactly four $0's$ while $25!$ ends with exactly six $0's$. And any number $n > 25$ will have that $n!$ has at least six $0's$. On the other hand, any number $m < 24$ will have that $m!$ has at most four $0's$.

(8)

By factorizing the left hand size we get that:

$$(x - y)(x + y) = 33 = 3 \times 11$$

Since the divisors of 33 are either 3×11 or 1×33. Then, we have two cases:

Case 1: $x - y = 3, x + y = 11$. This gives:

$$2x = 14 \rightarrow x = 7$$

$$y = 7 - 3 \rightarrow y = 4$$

$$(x, y) = (7,4).$$

Case 2: $x - y = 1, x + y = 33$. This gives:

$$2x = 34 \rightarrow x = 17$$

$$y = 33 - 17 \rightarrow y = 16$$

$$(x, y) = (17,16).$$

Solutions for gcd Exercises:

(1)

By factorizing both 8 and 9, we get that:

$$8 = 2^3, 9 = 3^2$$

And since they do not contain any common factors, we get that $gcd\ (8,9)\ = 1$.

(2)

By factorizing both 54 and 96, we get that:

$$54 = 2 \times 3^3, 96 = 2^5 \times 3$$

We take the common factors with the smaller power, we get that $gcd\ (54,96)\ = 2^1 \times 3^1$.

(3)

By factorizing both 35 and 91, we get that:

$$35 = 5 \times 7, 91 = 7 \times 13$$

We take the common factors with the smaller power, we get that $gcd\ (35,91)\ = 7^1$.

(4)

By factorizing both 54 and 6, we get that:

$$54 = 2 \times 3^3, 6 = 2 \times 3$$

We take the common factors with the smaller power, we get that $gcd\ (54,6)\ = 2^1 \times 3^1$.

(5)

By factorizing both 256 and 199, we get that:

$$256 = 2^8, 199 = 199$$

And since they do not contain any common factors, we get that $gcd\ (256,199)\ = 1$.

(6)

By factoring the number 120. We get that:

$$120 = 2^3 \times 3 \times 5$$

We will check which of the following choices and check which ones satisfy that $gcd\ (n, 120)\ = 24$:

(a) $gcd\ (120{,}2 \times 3^3)\ = 6$
(b) $gcd\ (120{,}2^2 \times 3^3)\ = 12$
(c) $gcd\ (120{,}2^3 \times 3^2 \times 11)\ = 24$
(d) $gcd\ (120{,}2^4 \times 3^3 \times 5)\ = 120$

Thus, the correct answer is (C).

Solutions for lcm Exercises:

(1)

By factorizing both 8 and 9, we get that:

$$8 = 2^3, 9 = 3^2$$

By considering all the factors with the higher power, we get that $lcm(8,9) = 2^3 \times 3^2$.

(2)

By factorizing both 54 and 96, we get that:

$$54 = 2 \times 3^3, 96 = 2^5 \times 3$$

By considering all the factors with the higher power, we get that $lcm(54,96) = 2^5 \times 3^3$.

(3)

By factorizing both 35 and 91, we get that:

$$35 = 5 \times 7, 91 = 7 \times 13$$

By considering all the factors with the higher power, we get that $lcm(35,91) = 5 \times 7 \times 13$.

(4)

By factorizing both 54 and 6, we get that:

$$54 = 2 \times 3^3, 6 = 2 \times 3$$

By considering all the factors with the higher power, we get that $lcm(6,54) = 2 \times 3^3$.

(5)

By factorizing both 256 and 199, we get that:

$$256 = 2^8, 199 = 199$$

By considering all the factors with the higher power, we get that $lcm(256, 199) = 2^8 \times 199$

.

(6)

By factoring the number 120. We get that:

$$120 = 2^3 \times 3 \times 5$$

We will check which of the following choices and check which ones satisfy that $lcm(n, 20) = 180$:

(a) $lcm\left(20, 2 \times 3^3\right) = 2^2 \times 3^3 \times 5 = 540$
(b) $lcm\left(20, 2^2 \times 3^2\right) = 2^2 \times 3^2 \times 5 = 180$
(c) $lcm\left(20, 2^3 \times 3^2\right) = 2^3 \times 3^2 \times 5 = 360$
(d) $lcm\left(20, 2^4 \times 3^3 \times 5\right) = 2^4 \times 3^3 \times 5 = 2160$

Thus, the correct answer is (b).

Solutions for Perfect Squares Exercises:

(1)

From the important rule, we get that:

$$n \times 18 = 6 \times 36 \rightarrow n = 12$$

(2)

By factorizing the numbers we get that:

- $196 = 2^2 \times 7^2$
- $192 = 2^6 \times 3$
- $169 = 13^2$
- $240 = 2^4 \times 3 \times 5$

Thus, the perfect squares are 169,196.

(3)

Notice that the sum of the digits of the number is:

$$0 \times 100 + 1 \times 100 + 2 \times 100 = 300.$$

From the divisibility rule by 3 and 9. We see that the number is divisible by 3 but not 9. Thus, the power of 3 that divides it is 1 which is odd. Therefore, the number cannot be a perfect square.

(4)

From the rule of the number of divisors. We have that the number of divisors of $p_1^{\alpha_1} \cdot p_2^{\alpha_2} \dots p_n^{\alpha_n}$ is:

$$(\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_n + 1)$$

If the number of divisors is odd. This means that all of the numbers in the brackets are odd. (Since if one of them is even we have $even \times odd = even$). This means that:

$$(\alpha_1 + 1) = odd, (\alpha_2 + 1) = odd, \dots, (\alpha_n + 1) = odd$$

And this means that all of α_k have to be even since $\alpha_k = odd - 1 = even$. Thus, the number is a perfect square.

(5)

Let us write the prime factorizations of both a, b. We have that:

$$a = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \dots p_n^{\alpha_n}, b = p_1^{\beta_1} \cdot p_2^{\beta_2} \dots p_n^{\beta_n}$$

Where $\alpha_k, \beta_k \geq 0$. Then, the rule of the greatest common divisor gives us:

$$gcd(a, b) = p_1^{min\,(\alpha_1,\beta_1)} \cdot p_2^{min\,(\alpha_2,\beta_2)} \dots p_n^{min\,(\alpha_n,\beta_n)}$$

on the other hand. The rule of the least common multiple gives us:

$$lcm(a, b) = p_1^{max\,(\alpha_1,\beta_1)} \cdot p_2^{max\,(\alpha_2,\beta_2)} \dots p_n^{max\,(\alpha_n,\beta_n)}$$

However, it is not hard to see that:

$$(\alpha_k, \beta_k) + (\alpha_k, \beta_k) = \alpha_k + \beta_k$$

(Since one of them is max and the other is the min)

Thus, we have:

$$d \times m = p_1^{min\,(\alpha_1,\beta_1)} \cdot p_2^{(\alpha_2,\beta_2)} \dots p_n^{(\alpha_n,\beta_n)} \times p_1^{max\,(\alpha_1,\beta_1)} \cdot p_2^{(\alpha_2,\beta_2)} \dots p_n^{(\alpha_n,\beta_n)}$$

$$= p_1^{min\,(\alpha_1,\beta_1)+max\,(\alpha_1,\beta_1)} \cdot p_2^{(\alpha_2,\beta_2)\,+max\,(\alpha_2,\beta_2)} \dots p_n^{(\alpha_n,\beta_n)\,+max\,(\alpha_n,\beta_n)}$$

$$= p_1^{\alpha_1+\beta_1} \cdot p_2^{\alpha_2+\beta_2} \dots p_n^{\alpha_n+\beta_n}$$

On the other hand, we have:

$$a \times b = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \dots p_n^{\alpha_n} \times p_1^{\beta_1} \cdot p_2^{\beta_2} \dots p_n^{\beta_n} = p_1^{\alpha_1+\beta_1} \cdot p_2^{\alpha_2+\beta_2} \dots p_n^{\alpha_n+\beta_n}$$

Therefore, we have $md = ab$ which is our desired result.

Solutions (**Combinatorics**)

Permutations with Repetition:

(1)

$$\frac{8!}{3! \times 2!} = 3360$$

(2)

$$\frac{10!}{4! \times 3! \times 2!} = 12600$$

(3)

$$\frac{9!}{3! \times 2!} = 30240$$

(4)

(a) $\frac{7!}{2!\times 2!} = 1260$

(b) $\frac{5!}{2!} = 60$

(5)

$$4! = 24$$

(6)

$$\frac{6!}{2!} = 360$$

(7)

The total number of arrangements without any restriction is $\frac{6!}{3!} = 120$

In all the words, the order of the letters A and B can be one of:

$$AAAB, AABA, ABAA, BAAA$$

(temporarily ignoring R and I).

Thus, in half of the cases, the B is in the middle. $\frac{120}{2} = 60$

(8)

The word $ELEMENTARY$ has 10 letters, with repetitions as follows:

$$E \times 3, T \times 1, L \times 1, M \times 1, N \times 1, A \times 1, R \times 1, Y \times 1$$

If we require the three E's to be together, we treat them as a single block (EEE).

That means the word now effectively consists of 8 distinct elements, so the number of arrangements is:

$$8! = 40320$$

(9)

We factor the number 231 into its prime factors: $231 = 3 \times 7 \times 11$. Each prime factor can be distributed among a, b, and c independently. For each prime factor, there are 3 choices; it can go to a, b, or c. Thus, the total number of ordered triples is: $= 3^3 = 27$ ordered triples.

Circular Permutations:

(10)

$$(5-1)! = 24$$

(11)

5 objects: $(5-1)! = 24$ ways.

7 objects: $(7-1)! = 720$ ways.

For n objects: $(n-1)!$ ways.

(12)

$$8! = 40{,}320$$

(13)

(a) $(36-1)! = 35!$

(b) $7! = 5040$

(14)

$(5-1)! = 24$

$5! = 120$

(15)

a) If flipping the necklace is not allowed (i.e., the two directions are considered different): $(13-1)! = 12!$

b) If flipping is allowed (i.e., a necklace and its reversed version are considered identical), we divide the result by 2, $\frac{12!}{2}$

(16)

We first arrange the doctors around the table. Since the seating is circular:

$(5-1)! = 24$ ways.

There are now 5 empty seats between the doctors where the engineers can sit.

We arrange the engineers in those seats:$5! = 120$ ways.

Therefore, the total number of arrangements is: $24 \times 120 = 2880$ ways

(17)

a) We first arrange the engineers around the table. Since the seating is circular: $(7-1)! = 720$ ways.

There are now 7 empty seats between the engineers where the doctors can sit.

We arrange the doctors in those seats

: ${}^{7}P_{5}$ ways.

Therefore, the total number of arrangements is: $720 \times {}^{7}P_{5}$ ways.

b) This is not possible because the number of empty seats between every two doctors is less than the number of engineers.

Combinations

(18)

$$^{5}C_{2} = 10$$

(19)

The number of letters other than E is seven, which can be arranged in $7!$ ways.

There are 8 gaps between them where the three E's can be placed, with no two E's adjacent. We choose 3 out of 8 positions (order doesn't matter).

$$7!.^{8}C_{3} = 282240$$

(20)

The case where the number of E's is greater than the number of F's means $E > 5$.

So, the number of E's can be $6, 7, 8, 9, or\ 10$.

The total number of ways is:

$$^{10}C_{6} + ^{10}C_{7} + ^{10}C_{8} + ^{10}C_{9} + ^{10}C_{10}$$

$210 + 120 + 45 + 10 + 1 = 386$ words.

(21)

Between the four letters: $^{4}C_{2} = 6$

Between the six odd numbers: $^{6}C_{2} = 15$

Between the three even numbers: $^{3}C_{2} = 3$

Total $= 6 + 15 + 3 = 24$ chords.

(22)

$a)\ {}^{8}C_{3}.{}^{5}C_{3} = 560$

$b)\ \frac{{}^{8}C_{4}.{}^{4}C_{4}}{2} = 35$

$c)\ \frac{{}^{8}C_{2}.{}^{6}C_{2}.{}^{4}C_{2}.{}^{2}C_{2}}{4!} = 105$

$d)\ \frac{{}^{8}C_{4}.{}^{4}C_{2}.{}^{2}C_{2}}{2!} = 210$

$e)\ {}^{8}C_{4}.{}^{4}C_{3}.{}^{1}C_{1} = 280$

(23)

The total number of people including Osama is 11.

Osama sits in a fixed position, with one friend on his right and another on his left.

The number of pairs of friends who can sit next to him = number of ways to choose 2 out of 10 friends:

${}^{10}C_{2} = 45$ different pairs.

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