NSMO

Mawhiba

Ministry of Education

National Science and Mathematics Olympiad

NSMO

Mathematics 2

General Administrations Competition

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Introduction

Our exceptional sons and daughters,

We are delighted to congratulate you on successfully completing the **Cities and Governorates stage** and qualifying for the **General Administrations stage**—an important, advanced step on your path toward mathematical challenge and innovation.

This **resource packet** is designed to **expand your understanding** across the four main branches of mathematics: **Combinatorics, Geometry, Algebra, and Number Theory**. We will focus on advanced concepts in **counting, geometric visualization, linear equations, and the principles of distribution and multiplication** in Number Theory.

This stage aims to **refine your skills in analytical thinking** and help you **connect mathematical concepts** to one another, applying them effectively in various problem situations.

This resource packet is a valuable opportunity to **deepen your understanding of mathematical patterns** and to use **logical reasoning** for justification and solving problems using organized methods.

We are confident in your abilities and look forward to seeing you **excel** in this crucial phase of your journey toward excellence.

The Scientific Team for the National Science and Mathematics Olympiad (NSMO) – Mathematics Track

First Unit: ALGEBRA

Percentages

Percent literally means "per hundred." Percent is just a shorthand way of writing the ratio of a number to 100.

For example, 54% means 54 out of 100 or

$$\frac{54}{100}$$

Specifically, if a is x percent of b, then

$$\frac{a}{b} = \frac{x}{100}$$

For example, 3 out of 16 is written as a percentage as follows

$$\frac{3}{16} = \frac{x}{100}$$

By simplifying, we obtain

$$x = 18\frac{3}{4}$$

Final

Answer: 3 is 18,75% of 16

- If a number, x , is increased by $k\%$, then the result is:

$$x \cdot \left(1 + \frac{k}{100}\right)$$

- Similarly, if x is decreased by $k\%$, then the result is:

$$x \cdot \left(1 - \frac{k}{100}\right)$$

Examples:

When a number is reduced by 40%, the result is 36. What is the original number?

Solution:

Let our number be x . We have

$$x\left(1-\frac{40}{100}\right)=36$$

$$\Rightarrow \frac{6}{10}x=36$$

$$\Rightarrow x=60$$

Exercises:

(1) For all nonzero x, y, and z, find the value of k such that:

$$\frac{7}{x+y}=\frac{k}{x+z}=\frac{11}{z-y}$$

(2) Find the values of $a, b, c \in N$ that satisfy the following:

$$\frac{a+1}{3}=\frac{b+2}{2}=\frac{8}{c+3}$$

(3) 36 is 120% of what number?

(4) What is the discount percentage for an item if its price changes from 2890 riyals to 2023 riyals?

(5) If Nora brought 60 cookies to school, gave 40% of them to her teachers, 25% of the rest to her friends, and ate one-third of what was left, how many cookies are left?

(6) If the ratio of the interior angles of a pentagon is $2 : 3 : 4 : 5 : 6$, what is the measure of the largest angle?

(7) If b is 5% greater than a, and b is 15% less than c,
What is the ratio of a to c ?

(8) A rectangle has been increased in length by 50% and in width by 20%. What is the percentage increase in its area?

(9) Every day, 20% of the fish are sold at the fish market. If 2000 fish remained at the end of Tuesday, how many fish were there at the beginning of Monday?

(10) If t is 25% of u , then what percent of $4t$ is $2u$?

(11) Given that

$yz : zx : xy = 1 : 2 : 3$ *and* $\frac{x}{yz} : \frac{y}{zx} = 1 : k$, Find k .

One-Variable Linear Equations

To solve a one-variable linear equation, follow the steps below in order:

- **Eliminating Denominators (if any)**

If there are fractions in the equation, multiply each term by the least common multiple (LCM) of the denominators to eliminate them.

- **Expand parentheses**

Use the distributive property to simplify parentheses, such as:

$$2(x+3) = 2x + 6$$

- **Move terms**

Move all the terms with the variable to one side and all the constants to the other using addition and subtraction.

- **Combine Like Terms**

Simplify both sides of the equation so that it takes the form:

$$ax = b \ \ , where\ a\ and\ b\ are\ constants.$$

- **Dividing by the variable's coefficient**

We divide both sides by a (the variable coefficient) to get the solution:

$$ax = b \Longrightarrow x = \frac{b}{a}$$

Note that:

- When $a \neq 0$ we have a unique solution:

$$x = \frac{b}{a}$$

- When $a = 0\ , b \neq 0$ there is no solution.
- When $a = 0\ , b = 0$ any real number is a solution to the equation.

- **Check your answer**

We can check our answer by substituting it back into the original equation. If the original equation is not satisfied by our answer, then we made a mistake.

- **Note:** Sometimes we do not follow the exact order of the previous steps when a more direct method is available.

Example:

Solve:

$$\frac{1}{10}\left\{\frac{1}{9}\left[\frac{1}{5}\left(\frac{x+2}{3}+8\right)+16\right]+8\right\}=1$$

Solution:

$$\frac{1}{10}\left\{\frac{1}{9}\left[\frac{1}{5}\left(\frac{x+2}{3}+8\right)+16\right]+8\right\}=1$$

$$\frac{1}{9}\left[\frac{1}{5}\left(\frac{x+2}{3}+8\right)+16\right]+8=10$$

$$\frac{1}{9}\left[\frac{1}{5}\left(\frac{x+2}{3}+8\right)+16\right]=2$$

$$\frac{1}{5}\left(\frac{x+2}{3}+8\right)+16=18$$

$$\frac{1}{5}\left(\frac{x+2}{3}+8\right)=2$$

$$\frac{x+2}{3}+8=10$$

$$\frac{x+2}{3}=2$$

$$x+2=6$$

$$x=4$$

Exercises:

(1) Solve:

$$1-\frac{x-\frac{1+3x}{5}}{3}=\frac{x}{2}-\frac{2x-\frac{10-6x}{7}}{2}$$

(2) If a, b, c are fixed positive numbers, solve the equation:

$$\frac{x-a-b}{c}+\frac{x-b-c}{a}+\frac{x-c-a}{b}=3$$

(3) Solve:

$$(x-3)^2+(x+1)^2+(4x-5)^2=0$$

(4) Solve:

$$ax+b-\frac{5x+2ab}{5}=\frac{1}{4}$$

(5) Given that the equation

$$a(2x+3)+3bx=12x+5$$

has infinitely many solutions for x. Find the values of $a$and b.

(6) Given that the equation

$$2a(x+6)=4x+1$$

has no solution, where a is a parameter, find the value of a

(7) Given that the equation

$$kx=12$$

has positive integer solution only, where k is an integer. Find the number of possible values of k.

(8) How many possible positive integer values of x satisfy the equation

$$\frac{1}{x}+\frac{1}{x+1}+\frac{1}{x+2}=\frac{13}{12}$$

(9) Given that the solution of equation

$$3a - x = \frac{x}{2} + 3$$

is 4. Find the value of $(-a)^2 - 2a$

(10) Solve the equation

$$\frac{x-n}{m} - \frac{x-m}{n} = \frac{m}{n}$$

(where $mn \neq 0$).

Multi-Variable Linear Equations

The general form of a system of two linear equations in two variables is:

$$\begin{cases} a_1x + b_1y = c_1 & (1) \\ a_2x + b_2y = c_2 & (2) \end{cases}$$

Where $a_1, b_1, c_1, a_2, b_2, c_2$ are real numbers

We have two linear equations in two variables, and they represent two straight lines in the coordinate plane.

The solution of the system corresponds to the point(s) of intersection of these two lines (since such a point satisfies both equations).

To eliminate one of the variables and solve the system, we use:

(i) the usual algebraic operations.

(ii) the substitution method.

In many cases, method (i) is more efficient.

When

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

- Thus, the two lines intersect at exactly one point, and therefore the system has a unique solution, which is

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \quad \square \quad y = \frac{a_1c_2 - a_2b_1}{a_1b_2 - a_2b_1}$$

When

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

- Then the two lines coincide, and therefore all their points are common. As a result, the system has infinitely many solutions

When

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

- Then the two lines are parallel and have no points of intersection; therefore, the system has no solution.

Example:

How many solutions does each of the following systems have?

$$(a)\ \begin{cases} 5x + 2y = 4 \\ 7x + 5y = 1 \end{cases}$$

$$(b)\ \begin{cases} 4x + 3y = 3 \\ 8x + 6y = 7 \end{cases}$$

Solution:

(a) One solution:

$$\frac{a_1}{a_2} = \frac{5}{7} \neq \frac{b_1}{b_2} = \frac{2}{5}$$

(b) There is no solution:

$$\frac{a_1}{a_2} = \frac{4}{8} = \frac{b_1}{b_2} = \frac{3}{6} \neq \frac{c_1}{c_2} = \frac{3}{7}$$

Exercises:

(1) How many solutions are there for this system?

$$\begin{cases} x + 5y = 3 \\ 2x + 10y = 6 \end{cases}$$

(2) Solve the following system of equations:

(a) $\begin{cases} x + y = 13 \\ x - y = 7 \end{cases}$ (b) $\begin{cases} 2x + y = 9 \\ x + 4y = 8 \end{cases}$

(c) $\begin{cases} x + y = 14 \\ x - y = 2 \end{cases}$ (d) $\begin{cases} 2x - y = 6 \\ x + 4y = 21 \end{cases}$

(3) Two numbers have a sum of 42 and a difference of 8. What are the two numbers?

(4) At a certain time, Reema notices that her digital watch reads a minutes after two o'clock. Fifteen minutes later, it reads b minutes after three o'clock. She is amused to note that a is six times b. What time was it when she looked at her watch for the second time?

(5) Solve the following system of equations:

$$\begin{cases} \dfrac{x-y}{5} - \dfrac{x+y}{4} = \dfrac{1}{2}, \\ 2(x-y) - 3(x+y) + 1 = 0 \end{cases}$$

(6) Solve the following system of equations:

$$\begin{cases} 5.4x + 4.6y = 104 \\ 4.6x + 5.4y = 96 \end{cases}$$

(7) Solve the following system of equations:

$$\begin{cases} (x+2)(5x+y) = 16 \\ 5x + y = 7 \end{cases}$$

(8) Solve the following system of equations:

$$\begin{cases} \dfrac{x}{2} = \dfrac{y}{3} = \dfrac{z}{5} \\ x + 3y + 6z = 15 \end{cases}$$

(9) Solve the following system of equations:

$$\begin{cases} x + 2y = 5 \\ y + 2z = 8 \\ z + 2u = 11 \\ u + 2x = 6 \end{cases}$$

(10) Solve the following system of equations:

$$\begin{cases} 5x - y + 3 = a \\ 5y - z + 3x = b \\ 5z - z + 3y = c \end{cases}$$

(11) Given that $x = 2, \quad y = 1$ is the solution of system

$$\begin{cases} ax + by = 7, \\ bx + cy = 5 \end{cases}$$

What is the relation between a, c ?

(12) Find the value of (a, b, c) if both systems have the same solution

$$\begin{cases} 3x - y = 5 \\ 2x + y - z = 0 \\ 4ax + 5by - z = -22 \end{cases} \quad , \quad \begin{cases} ax - by + z = 8 \\ x + y + 5 = c \\ 2x + 3y = -4 \end{cases}$$

(13) Determine the values of k such that the system of equations

$$\begin{cases} kx - y = -\frac{1}{3} \\ 3y = 1 - 6x \end{cases}$$

$a)$ has unique solution $\quad b)$ no solution $\quad c)$ infinitely many solutions

(14) Given that x, y, z satisfy the system of equations

$$\begin{cases} 2010(x - y) + 2011(y - z) + 2012(z - x) = 0 \\ 2010^2(x - y) + 2011^2(y - z) + 2012^2(z - x) = 2011 \end{cases}$$

Find the value of $z - y$

(15) Solve the following system of equations:

$$\begin{cases} x - y - z = 5 \\ y - z - x = 1 \\ z - x - y = -15 \end{cases}$$

(16) Solve the following system of equations:

$$\begin{cases} x - y + z = 1 \\ y - z + u = 2 \\ z - u + v = 3 \\ u - v + x = 4 \\ v - x + y = 5 \end{cases}$$

Challenge Problems:

(1) Solve the following system of equations:

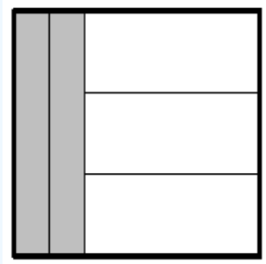
$$\begin{cases} \frac{1}{a} - \frac{1}{b} = 6 \\ \frac{1}{a} + \frac{1}{b} = 14 \end{cases}$$

(2) Solve the following system of equations:

$$\begin{cases} x + y = 8 \\ y + z = 4 \\ z + x = 6 \end{cases}$$

(3) Give an example of two fractions whose difference is equal to their product.

(4) The figure shown is a square composed of five rectangles, all having the same perimeter.
What is the ratio of the area of one shaded rectangle to the area of one unshaded rectangle?



(5) We want to divide a certain amount of money equally among a group of children. If each child receives 60 halalas, 2.10 riyals will remain. However, if each child receives 20 halalas more than this amount, there will be just enough for each child to receive 70 halalas. How many children are in the group?

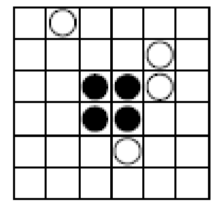
(6) The number 3600 can be written as $2^a \times 3^b \times 4^c \times 5^d$. where $a, b, c, and\ d$ are positive integers.
If $a + b + c + d = 7$. what is the value of c ?

(7) If you can remove ten digits from the number 12345123451234512345123451234512345. What is the largest number you can get?

(8) Cut the following shape into four identical parts

(both in shape and area)

so that each part contains one black circle and one white circle.



(9) My brother has four children. They are $5, 8, 13, and\ 15$ years old. Their names (in no particular order) are Muhammad, Raja, Najah, and Nour. One of the girls is in kindergarten, Raja is older than Muhammad, and the sum of Raja and Nour's ages is divisible by three.

Is Nour a boy or a girl?

(10) There are one hundred people living on an island. Some of them always lie, while others always tell the truth, and each of them has a favorite season of the year. Every person was asked the following four questions:

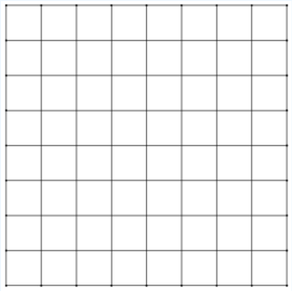
- **Do you like winter?**
- **Do you like spring?**
- **Do you like summer?**
- **Do you like fall?**

There were 25 "yes" answers to the first question, 25 "yes" answers to the second, 45 "yes" answers to the third, and 55 "yes" answers to the fourth.

How many liars are there on the island?

(11) We have the following 8×8 table divided into 64 small squares.

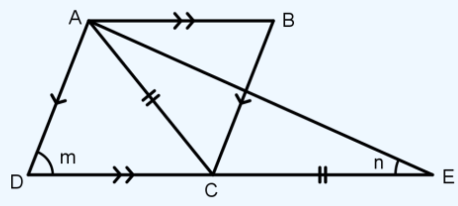
Can you color 17 small squares black so that no two squares share an edge or even a corner?



Second Unit: Geometry

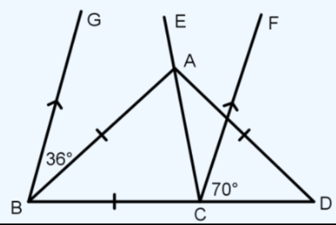
Revision Exercises

(1) In the following diagram $ABCD$ is a rhombus, ACE is an isosceles triangle with $AC = CE, D, E, C$ are on the same line. If $\angle AEC = n, \angle ADC = m$. Write an equation the relates m, n.

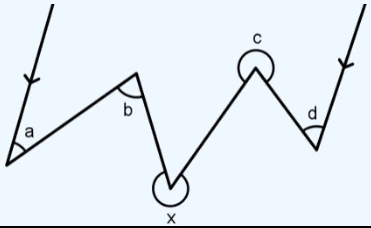


(2) In the following diagram $ABCD$ is a quadrilateral. Point E lies inside such that $\angle ABE = \angle DBC = x, \angle ACE = \angle DCB = y$. If $\angle BEC = 125°, \angle BDC = 78°$ Find $\angle A$.

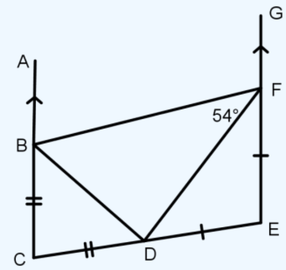
(3) In the following diagram: given $\underline{AB} = \underline{BC} = \underline{AD}$ Find $\angle EAD$.



(4) Find the value of x in terms of a, b, c, d.



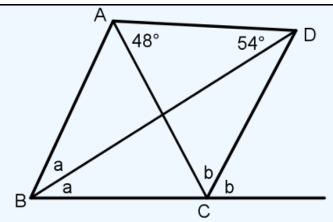
(5) In the following diagram we have $\underline{BC} = \underline{CD}, \underline{DE} = \underline{EF}, AC \parallel GE$. Given $\angle BFD = 54°$ Find $\angle DBF$.



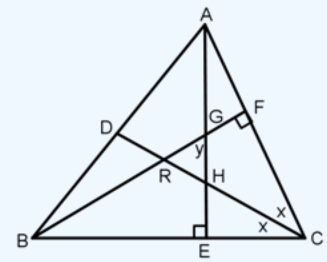
(6) Given the information in the diagram:

(a) Find $\angle BAC, \angle BDC$ in terms of a, b.

(b) if $a = \frac{2}{3}b$. Find a, b.

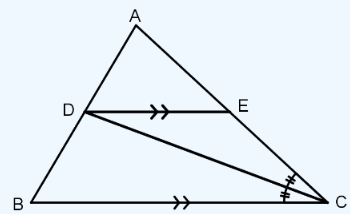


(7) In the following diagram: give an equation that relates x, y.

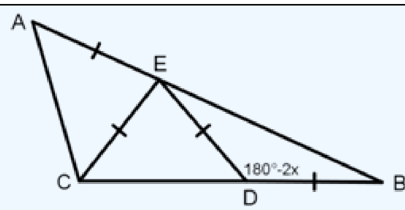


(8) in the following diagram: CD bisects $\angle ACB, \angle ACB = 40°, \angle B = 70°$,

$DE \parallel BC$. Find the measure of $\angle EDC, \angle BDC$.

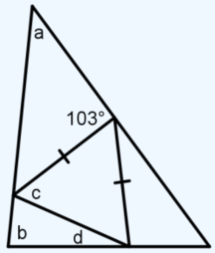


(9) In triangle ΔABC, E, D lie on $\underline{AB}, \underline{BC}$ such that $\underline{AE} = \underline{EC} = \underline{DE} = \underline{DB}$. If $\angle EDB = 180° - 2x$. Find $\angle ACE$ in terms of x.



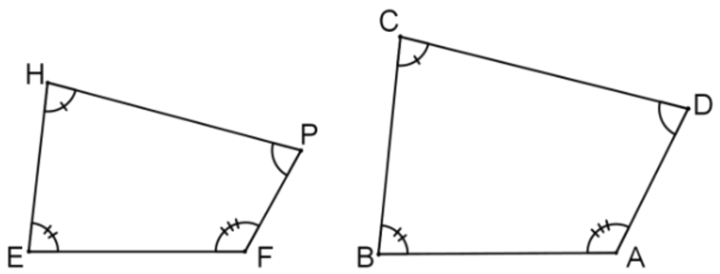
(10) In the following figure:

If $a = \frac{2}{3}c = \frac{1}{2}b = 3d$. Then find . a, b, c, d.



similarity

Definition 1:



Any two polygons are **similar** if the two following conditions are met:

(1) The corresponding angles are equal in measure.

(2) The lengths of the corresponding sides are proportional

So, in the polygons above, if we have:

$$\left\{\angle A = \angle F, \angle B = \angle E, \angle C = \angle H, \angle D = \angle P, \frac{AB}{FE} = \frac{BC}{EH} = \frac{CD}{HP} = \frac{DA}{PF}\right\} \Longrightarrow ABCD \sim FEHP$$

Theorem 1 :

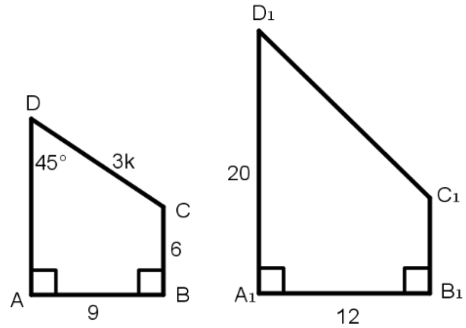
The ratio between the perimeters of similar polygons is equal to the ratio of any two corresponding sides

Exercises:

In exercises $(1-8)$ choose the appropriate answer (only one is correct):

Exercise	Polygons	Cannot be similar	Can be similar	Always similar
(1)	**Equilateral** triangles			
(2)	**Isosceles triangles**			
(3)	**Squares**			
(4)	**Rhombuses**			
(5)	**Right triangles**			
(6)	**Scalene Triangles**			
(7)	**Rectangles**			
(8)	**Right triangle and acute triangle**			

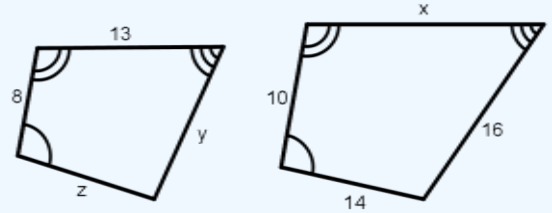
Exercises $(9-16)$ are on the diagrams below. Where quadrilaterals $ABCD \sim A_1B_1C_1D_1$ are similar:



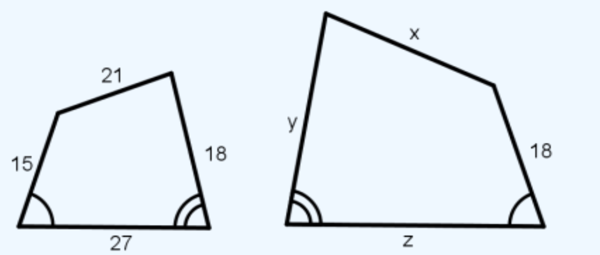
(9)	Similarity ratio of $ABCD$, $A_1B_1C_1D_1$.	
(10)	Type of quadrilateral $A_1B_1C_1D_1$	
(11)	Measure of $m\angle D_1$	
(12)	Measure of $m\angle C_1$	
(13)	Measure of C_1B_1	
(14)	Measure of AD	
(15)	Measure of C_1D_1	
(16)	Ratio of the perimeters of the quadrilaterals	

Find the values of x, y, z if the two polygons in each exercise are similar.

(17)



(18)



Triangle Similarity

Theorem (2):

For any two similar triangles we have:

- The corresponding sides are proportional.
- The corresponding altitudes are proportional and have the same ratio as the similarity ratio.
- The corresponding medians are proportional and have the same ratio as the similarity ratio.

Similarity Postulate (AAA)

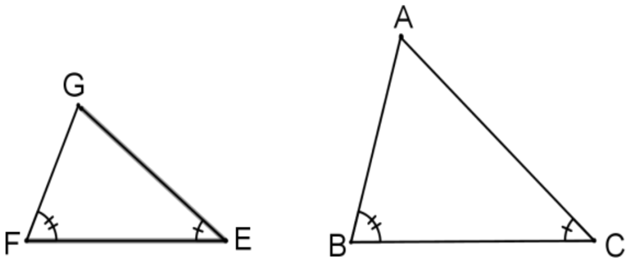
Theorem 3:

Similarity Postulate: AAA

Two triangles are similar if their corresponding angles are equal in measure. We call this case (AAA) (Angle- Angle- Angle). Of course, it suffices to show similarity using two angles instead of all three since the sum of angles in a triangle is 180^o.

In the adjacent triangles $\Delta ABC, \Delta GFE$, if

$$\angle B = \angle F, \angle E = \angle C.$$



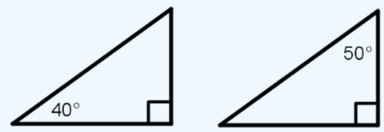
Then the triangles are similar and we get that:

$$\frac{AB}{GF} = \frac{BC}{FE} = \frac{CA}{EG}.$$

Exercises:

In $(1-9)$, determine the similar triangles or state that there aren't any if you cannot find any.

(1)



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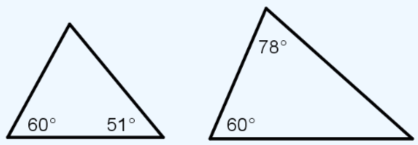
(2)



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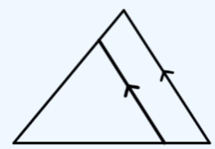
(3)



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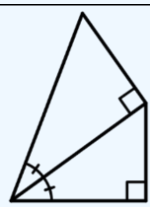
(4)



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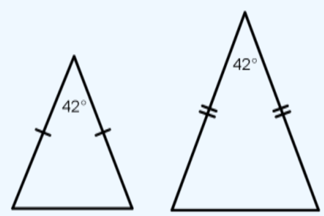

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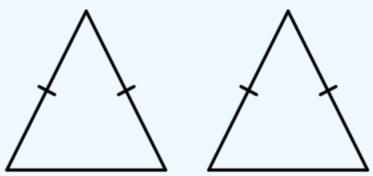

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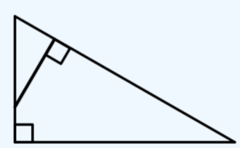

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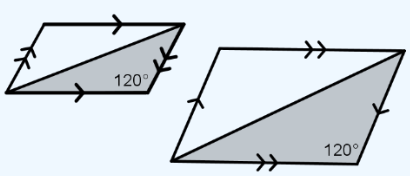

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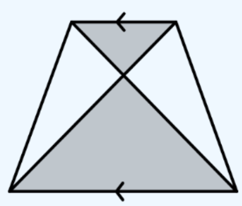

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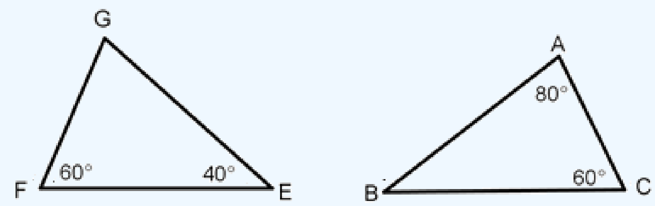
(10) Look at the adjacent figure, and answer the following:

$\angle ABC =$ ______

$\Delta ABC \sim$ ______

$\frac{AB}{.....} = \frac{BC}{.....} = \frac{AC}{.....}$

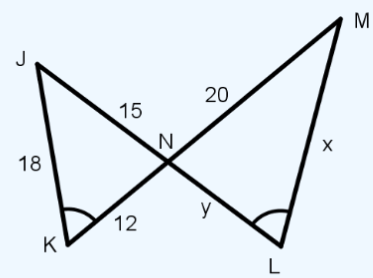


(11) Look at the adjacent figure, and answer the following:

(a) $\Delta JKN \sim \dots\dots\dots\dots\dots..$

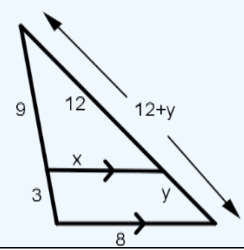
(b) $\frac{15}{.....} = \frac{12}{.....}$, $\frac{15}{.....} = \frac{18}{.....}$

(c) $x = \cdots$, $y = \cdots$



Find the values of x, y in the following exercises:

(12)



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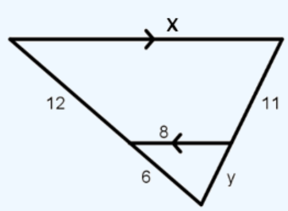

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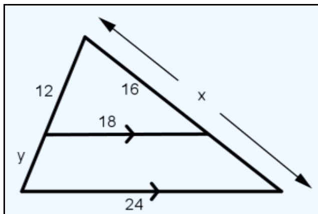
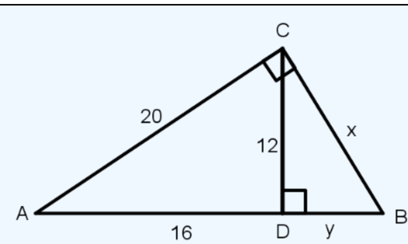


(15) In the adjacent figure:

a) Give two triangles that are similar to ABC.

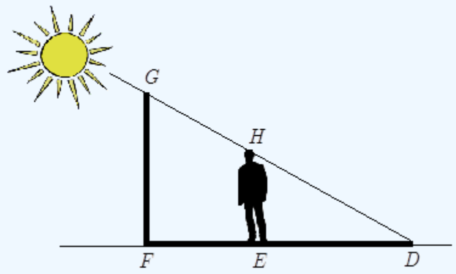
b) Find the value of x, y.



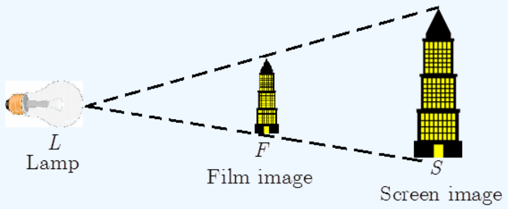
(16) To approximate the height of the basketball stand, one of the players whose height is exactly $2m$ stood so that his shadow ends with the end of stand's shadow. We found that:

$\underline{DE} = 1.6,\ \underline{DF} = 4.4$

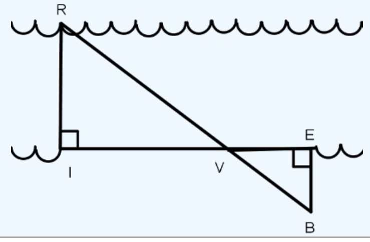
Find the height of the stand.



(17) In the adjacent figure, a picture of a building appears on the film as well as the screen (light source projects the film image on the screen). If $LF = 6cm, LS = 24m$, and the building height on the screen was $2.2m$. What is the height of the building on the film?



(18) In the adjacent figure, If $IV = 63m, VE = 20m, BE = 15m$, Find the width of the river.



Similarity Postulates $(SAS), (SSS)$

Theorem 4:

Two triangles are similar if a corresponding angle is equal, and the ratio of the two corresponding sides that form the angle are equal. We call this case $(S.A.S)$ (side- angle- side).

Theorem 5:

Two triangles are similar if the ratio of the three corresponding sides are equal. We call this similarity $(S.S.S)$ (side-side-side).

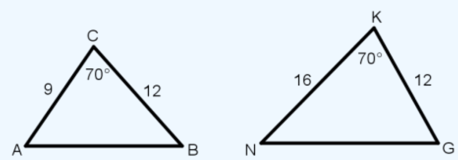
Then, for any two similar triangles we have:

1. Corresponding sides are proportional.
2. Corresponding altitudes are proportional and have the same ratio as the similarity ratio.
3. Corresponding medians are proportional and have the same ratio as the similarity ratio.
4. Corresponding angle bisectors are proportional and have the same ratio as the similarity ratio.
5. The ratio of their perimeters is equal to the similarity ratio.
6. The ratio of their areas is the square of the similarity ratio.

Exercises:

In $(19 - 24)$, determine the similar triangles, and the similarity postulate that you used.

(19)



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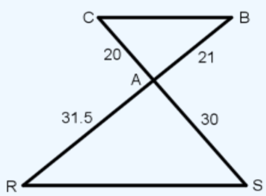

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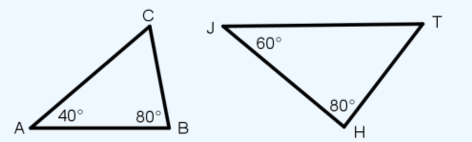

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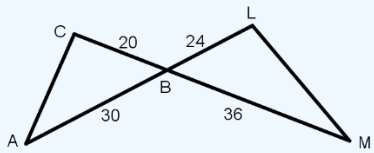

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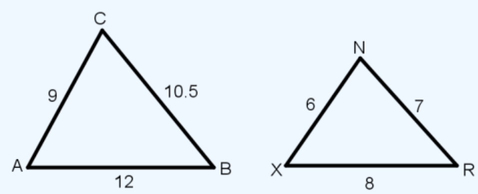

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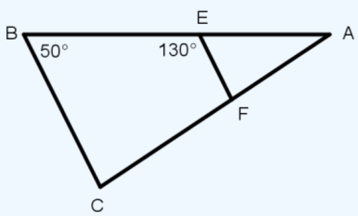

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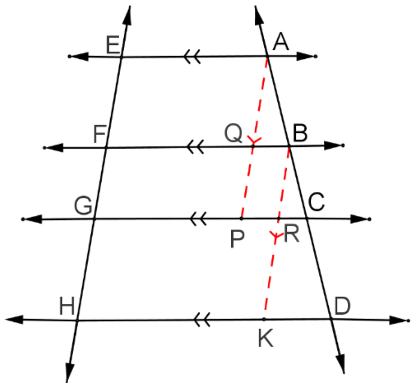
Proportional Lengths

Theorem 6: (Thales Theorem)

If two lines are intersected by several parallel lines, the segments formed on one transversal are proportional to the segments formed on the other transversal.



Proof: Here, we have $AE \parallel BF \parallel GC \parallel HD$, and are intersected by lines AD, EH. We need to show that:

$$\frac{AB}{EF} = \frac{BC}{FG} = \frac{CD}{GH}$$

To prove this theorem, we draw AP and BK parallel to EH as shown in the adjacent figure. Since:

$AQ \parallel FE$ and $QP \parallel GH$, and $BR \parallel FG$ and $RK \parallel GH$ are parallels to sides,

Then, $AQ = EF, BR = QP = FG, RK = GH.$

In $\triangle APC, BQ \parallel PC$, hence:

$$\frac{AB}{BC} = \frac{AQ}{QP}, \frac{AB}{BC} = \frac{EF}{FG} \Rightarrow \frac{AB}{EF} = \frac{BC}{FG}.$$

Similarly, in $\triangle BKD, BQ \parallel PC$, we obtain:

$$\frac{BC}{CD} = \frac{BR}{RK} \Rightarrow \frac{BC}{FG} = \frac{CD}{GH}.$$

Hence:

$$\frac{AB}{EF} = \frac{BC}{FG} = \frac{CD}{GH}.$$

Theorem 7:

If a line is drawn parallel to one side of a triangle and intersects the other two sides, it divides them into segments proportional in length.

Theorem 8: (Angle Bisector Theorem)

If the vertex angle of a triangle (or its external angle) is bisected internally or externally, the ratio of the segments of the base equals the ratio of the lengths of the other two sides.

Proof:

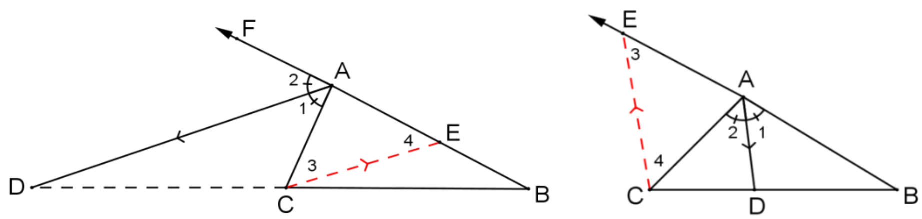


Figure1 Figure 2

In figure 1: Ray AD Bisects $\angle CAB$ and intersects BC at D.

In figure 2: Ray AD Bisects $\angle CAF$ and intersects BC at D.

We need to prove:

$$\frac{AB}{AC} = \frac{BD}{DC}.$$

To prove this, draw $CE \parallel AD$ intersecting AB at E. Then:

$$\angle 1 = \angle 3(corresponding), \angle 2 = \angle 4(alternate).$$

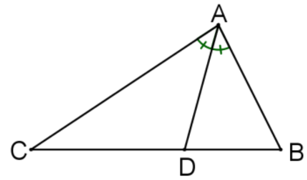
Thus: $\angle 3 = \angle 4$. Therefore $AE \parallel AC$. Since $CE \parallel AD$, we have:

$$\frac{AB}{AE} = \frac{BD}{DC}.$$

Hence:

$$\frac{AB}{AC} = \frac{BD}{DC}.$$

Notes:

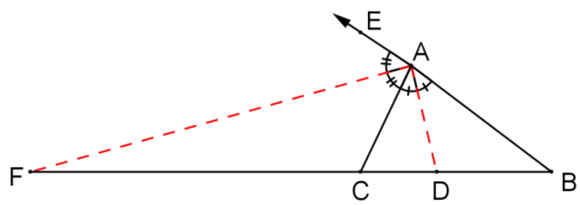


1. The converse of the theorem holds: if D divides BC such that

$$\frac{AB}{AC} = \frac{BD}{DC}$$

then AD bisects $\angle A$.

2. AF and AD are the internal and external bisectors of $\angle A$.



Hence:

$$\frac{AB}{AC} = \frac{BD}{DC}, \frac{AB}{AC} = \frac{BF}{FC}$$

thus

$$\frac{BF}{FC} = \frac{BD}{DC}.$$

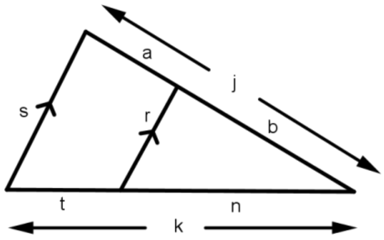
Important Note:

The measure of the angle between the internal and external bisectors of the same angle is 90°.

Exercises:

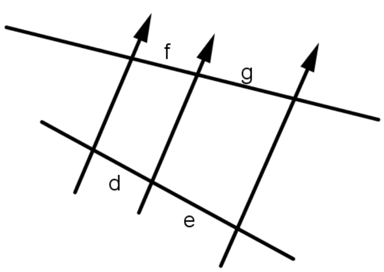
(1) Look at the following figure, and decide which of the relations are correct.

Part	Relation	Answer
(a)	$\frac{r}{s}=\frac{a}{b}$	
(b)	$\frac{t}{k}=\frac{a}{j}$	
(c)	$\frac{j}{a}=\frac{s}{r}$	
(d)	$\frac{r}{s}=\frac{n}{k}$	
(e)	$\frac{a}{b}=\frac{n}{t}$	
(f)	$\frac{b}{j}=\frac{t}{k}$	



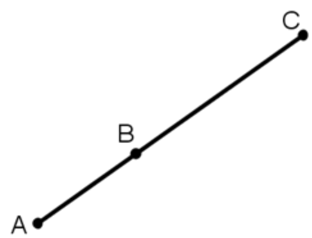
(2) Look at the following figure, and decide which of the ratios are correct.

Part	Relation	Answer
(a)	$\frac{d}{f}=\frac{g}{e}$	
(b)	$\frac{f}{g}=\frac{e}{d}$	
(c)	$\frac{g}{f}=\frac{e}{d}$	
(d)	$\frac{d}{f}=\frac{e}{g}$	



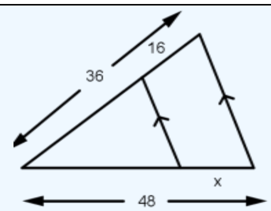
In $(3-6)$, if $\frac{AB}{BC}=\frac{3}{5}$, complete the other entries of the table.

	(3)	(4)	(5)	(6)
AB	6			
BC		25		
AC			56	100



In $(7-15)$, find the values of x, y.

(7)



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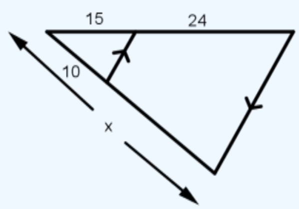

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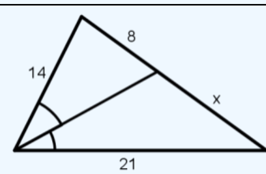

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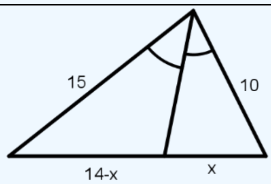

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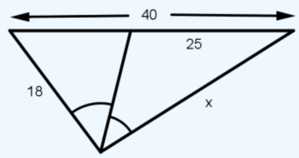

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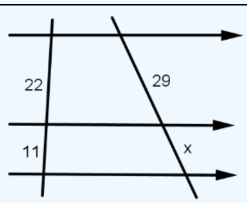

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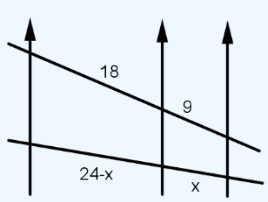

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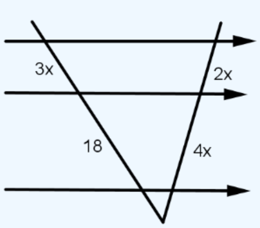


Third Unit: Number Theory

Even and Odd Integers

Integers can be divided into two types of numbers: those that are divisible by 2 are called **even** numbers, and the rest of the numbers that are not divisible by 2 are called **odd** numbers. An even integer can be written in the form $2k$, where k is an integer. An odd integer can also be written in the form $2k + 1$, where k is an integer. Any given integer is exclusively either even or odd; it cannot be both.

Other Properties of Even and Odd numbers:

(1) Odd ≠ Even.

(2) Odd + Even = even +odd = odd, odd-even = even – odd = odd.

(3) even ± even= odd± odd= even.

(4) If the multiplication of two integers is even, then one of them has to be even.

(5) The multiplication of two consecutive integers has to be even.

(6) If the sum (or difference) of some integers is odd, then the number of odds in those numbers is odd.

(7) If the sum (or difference) of some integers is even, then the number of odds in those numbers is even.

(8) if the multiplication of some integers is odd, then all of those integers must be odd.

(9) if the multiplication of some integers is even, then at least one of those integers must be even.

Example 1:

Divide the following numbers into even and odd: $102, 203, 519, 3340, 70015, 87654$. Then find a pattern for even and odd numbers.

Solution:

- The number 102 is even since we can write it as $102 = 2(51)$.
- 203 is odd since $203 = 2(101) + 1$.
- 519 is odd since $519 = 2(259) + 1$.
- 3340 is even since $3340 = 2(1670)$.
- $70015 = 2(35007) + 1$ so it is an ___ number.
- $87654 = 2(43827)$ so it is an ___ number.

If the units digit is $0,2,4,6,8$ then the number is ___.

If the units digit is $1,3,5,7,9$ then the number is ___.

Example 2:

If we add two even numbers, then the result would be:

(a) Even (b) Odd

Solution:

Answer is (a) Since we know from the properties that the sum of two even numbers is even. Moreover, we can check that the units digit if we add two of $0,2,4,6,8$ then the resulting number would still have a $0,2,4,6,8$ as a units digit.

Can you use the same logic to determine the following:

If we add two odd numbers, then the result would be:

(a) Even (b) Odd

If we add an even and an odd number, then the result would be:

(a) Even (b) Odd

Example 3:

If we pick three random numbers a, b, c then the numbers:

$$\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2}$$

(a) all are non-integers (b) at least one of them is an integer.

(c) At least two of them are integers (d) All of them are integers.

Solution:

The answer is (b). We can see that since we have two cases:

Case 1: All of the numbers have the same parity (all odd or all even), which means that sum of any two is even so it is divisible by 2. This leads to $\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2}$ being all integers.

Case 2: One of them has a different parity (one odd and two even or one even and two odd). And if we do this case (For example a, c are odd while b is even). We can see that two sums ($a + b, b + c$ are odd, while $a + c$ is even). Thus, only one of $\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2}$ is an integer. Following the same logic. The other case should also result in one number being an integer, which gives the answer (b).

Example 4:

If you know that the sum of 100 positive integers is 10000, and that the number of odd numbers (in the 100 numbers) is more than the number of even numbers. Find the largest number of even numbers in the set.

Solution:

Assuming that the number of even numbers is x and that the number of odd numbers is y. Then we have:

$$x + y = 100, y > x$$

But since we know that the sum is even, so y has to be even (look at the properties of even and odds). Thus, y is at least 52, and that means that x is at most 48. It suffices to give an example which is: $\underbrace{1 + 1 + \ldots + 1}_{52 time} + \underbrace{2 + 2 + \ldots + 2}_{47 time} + 9854 = 10000$

Thus, the answer is 48.

Example 5:

We have n numbers: $x_1, x_2, x_3, \ldots, x_n$ all of them are either $1, -1$. If:

$$x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n + x_nx_1 = 0$$

Then what can we say about n?

(a) even (b) odd (c) n is multiple of 4 (d) cannot say anything about n.

Solution:

The answer is (c). Notice that the sum:

$$x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n + x_nx_1$$

Contains n terms each of which is either 1 or -1. Thus, n has to be even so that the sum is 0. Now we can write $n = 2k$, this means that the numbers $x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1$ have k of them equal to 1 while the other k are equal to -1. But notice that their multiplication is:

$$(x_1x_2).(x_2x_3), \ldots .(x_nx_1) = x_1^2x_2^2 \ldots . x_n^2 = 1$$

Thus:

$$(-1)^k(1)^k = 1$$

This means that k should also be even. Therefore, $k = 2m \Rightarrow n = 2k = 4m$ which gives us that n is a multiple of 4.

Exercises:

1) Find the parity of the number $\frac{12214509875435678863334542149 38}{2}$ without finding the value of the number.

2) We have 50 books that we would like to put in 5 boxes. Can we divide the books into the boxes so that each box has an odd number of books? Explain your answer

3) If you knew that a, b are two consecutive integers, $c = ab, N^2 = a^2 + b^2 + c^2$. Determine the parity of the number N (odd or even) and show your reasoning.

4) Challenge: in the following pattern $1,2,5,13,34,89, \ldots$. in any three consecutive numbers, the sum of the first and third numbers it is equal to three times the number in the middle. For example, $1,2,5$ we have $1 + 5 = 3(2)$. Or $2,5,13$ we have $2 + 13 = 3(5)$ and so on. What is the parity of the 2003^{th} number?

5) Given three integer numbers x, y, z two of which are odd and the third is even. Show that:

$$(x + 1)(y + 2)(z + 3)$$

Has to be even.

6) If you can put wither $' + '$ or $' - '$ signs between any two numbers in the list:

$$1 \quad 2 \quad 3 \quad 4 \quad \ldots\ldots \quad 2017$$

Will the result be positive or negative?

7) (a) we have the line

$$1 * 2 * 3 * 4 * 5 * 6 * 7 * 8 = 0$$

Can we replace the stars with $+, -$ to make the statement correct?

(b) Same questions but for the statement:

$$1 * 2 * 3 * 4 * 5 * 6 * 7 * 8 * 9 = 0$$

8) In a chess competition, each player will play 20 games. Wins add 3 points to the total score, draws add one point to the score, and defeats subtract one point from the total score. Can a player get a final score of 39? Show your reasoning.

9) (a) There are 10 baskets on a circle. Can we put oranges in each basket so that the difference in the number of oranges between each consecutive baskets is 1?

(b) What if there was 9 baskets instead? Show your reasoning.

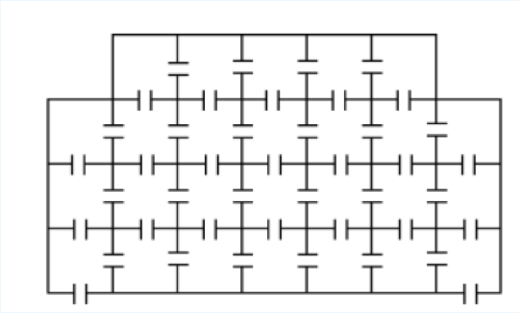
10) A worm move on a straight line, she can jump 6 or 8 cm in any of the directions (left or right). Can the worm reach to a point that is:

(a) $1.5\ cm$ (b) 7 cm (c) $4\ cm$

Far away from its original position?

11) Challenge: how can we arrange the 10 numbers 1,1,2,2,3,3,4,4,5,5 in a line so that there is one number between 1,1, two numbers between 2,2, three numbers between 3,3, four numbers between 4,4, and 5 numbers between 5,5?

12) Challenge: How can we visit the 26 rooms in the diagram such that we enter each room exactly one time?



Prime and Composite Numbers

Revision

- Between all the positive integers, the number 1 is the only number that has 1 divisor which is itself.
- All positive integers larger than 1 have at least two positive divisors.
- If a positive integer has only two positive divisors (itself and 1), then we call it a **prime** number.
- All positive integers larger than 1 and are not primes are called **composite** numbers.

From the previous definitions, we conclude the following:

- The number 1 is neither a prime nor a composite number.
- There is only one even prime number that is 2. And it is the smallest prime.
- The smallest composite number is 4.

All positive numbers N greater than 1 can be factorized into its prime factors. That is:

$$N = p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3}.....p_k^{\alpha_k} \text{ Such that } p_1, p_2, p_3, \ldots\ldots, p_k$$

Are different primes. And $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_k$ are non-negative numbers. Then, the number of divisors of N is:

$$(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)\ldots.(\alpha_k + 1).$$

Example 1:

If $p, p^3 + 5$ Then $p^5 + 7$ is:

(a) Prime (b)composite (c) Prime or composite (d) Not A prime nor a composite

Solution:

(b) composite number.

Notice that if p is odd then $p^3 + 5$ is an even number larger than 2 and that makes a composite number. Thus $p = 2$, which makes $p^5 + 7 = 2^5 + 7 = 39 = 13 \times 3$ which is a composite number.

Example 2:

Given three prime numbers p, q, r which satisfy the condition

$$p + q = r, p < q$$

Find the value of p.

Solution:

Notice that r cannot be even since the only even prime is 2 and it cannot be the sum of two even primes. Then, r is odd, which makes p, q one even and one odd number. However, we know that the only even prime is 2 . Moreover, $p < q$, which leads to $p = 2$.

Exercises:

1) Given x, y are prime numbers. Find the number of ordered solutions such that $x + y = 75$.

(a) 1 (b) 2 (c) 3 (d) 4

2) We call the two digit prime number $\underline{ab}$ a naughty number if $\underline{ba}$ is also prime. Find the number of naughty prime numbers.

3) Given the different prime numbers a, b, c that satisfy the equation $ab^b c + a = 2000$. Find all possible values of $a + b + c$.

4) Given that $p, q, p - q$ are primes, and $p + q$ is even. Find the value of:

$$(1 + \frac{1}{2})^p (1 - \frac{1}{3})^q$$

5) Challenge: If you knew that n is a positive integer such that $n + 3, n + 7$ are both primes. Find the remainder of n when divided by 3.

6) Challenge: if p is a prime that not less than 5, and $2p + 1$ is also a prime. Prove that $4p + 1$ is a composite.

7) We have the equation $56a = 65b$ with a, b positive integers. Prove that $a + b$ is a composite number.

8) We have $m > n$ two positive numbers that also satisfy $m^2 - n^2 - 2m - 2n = 19$. Find the value of m, n.

9) Given three primes p, q, r that satisfy $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1661}{1986}$ find $p + q + r$.

10) Given m, n are two different primes. And p is the smallest integer that satisfies $p = m + n + mn$. Find the value of $\frac{m^2+n^2}{p^2}$.

11) Given p, q are primes, and $p = m + n, q = mn$ such that m, n are positive integers. Find the value

$$\frac{p^p + q^q}{m^n + n^m}$$

12) Given p, q primes that satisfy the equation $5p + 7q = 129$. Find the values of $p + q$.

13) Challenge: Given $p, p + 2, p + 6, p + 8, p + 14$ are all prime numbers. Find the value of p.

14) Given p, q are two consecutive primes. The integer n satisfies that:

$$\{n - 1, 3n - 19, 38 - 5n, 7n - 45\} = \{p, 2p, q, 2q\}$$

Not necessarily in this order. Find the value of n.

15) Challenge: Prove that there exists an infinite number of values for n such that $n^2 + n + 41$ is a composite number.

Fourth Unit: Combinatorics

Counting Principles

Before diving into the next counting principles, it's important to review the concepts we discussed earlier.

Review: The Two Basic Counting Principles

Multiplication Principle: If event A can occur in m different ways, and event B can occur in n different ways, and the two events are independent, then both events together can occur in $m \times n$ ways.

Addition Principle: If event A can occur in m different ways, and event B can occur in n different ways, and the two events are mutually exclusive, then A or B can occur in $m + n$ ways.

Exercises:

(1) A restaurant offers meat sandwiches in three sizes (small – medium – large), with cheese, tomato, or without either. Faisal wants to order a cheese sandwich. In how many ways can he do that?

(2) A shoe store has 6 types of shoes, and each type comes in 7 colors. How many different shoes are available in total?

(3) Salman, Osama, and Khalid are in a race. In how many ways can the race end if two or more of them can tie?

(4) The National Bank requires new customers to choose a 4-digit PIN using distinct digits from 1 to 5, with the condition that the absolute difference between any two consecutive digits is at least 2. In how many ways can the customer choose the PIN?

(5) In how many ways can 3 different math books, 4 different chemistry books, and 5 different physics books be arranged on a shelf if books of the same subject must be grouped together?

First: Counting Numbers and Digit Strings

- **A natural number with n digits:** We say that a natural number consists of one or more digits (places) if its leftmost nonzero digit is not zero.

For example: The number *2345* has four digits, but the number *034* is not considered a three-digit number because the leading zero doesn't count. It's written as *34*, meaning the number actually has two digits.

- **A string of digits:** A digit string is made up of one or more digits, regardless of whether the leftmost digit is zero or not.

For example: each of the following: *01234*, *00034*, and *12302* is a string made up of five digits.

- **Palindromic numbers:** A natural number is called palindromic (symmetric) if its digits read the same from right to left and from left to right.

For example:

1234321, *919*, and *6* are palindromic numbers, while *2342* is not.

The two counting principles will help us find the number of required numbers or strings in a simple yet deep way. Let's begin our creative journey!

Exercises:

(6) How many positive integers with four different digits are there?

(7) How many positive odd integers consist of five digits?

(8) How many two-digit numbers can be formed from the digits of the set $\{0,3,4,5,6,7,8,9\}$, provided that the sum of their digits is odd?

(9) How many numbers between 0 and 1000 contain the digit 5 exactly once?

(10) How many three-digit numbers that are divisible by 3 and have distinct digits can be formed from the digits $\{1, 3, 7, 8, 9\}$?

(11) How many 10-digit strings consisting of only 0s and 1s contain exactly five consecutive zeros?

(12) How many five-digit numbers satisfy the condition that the sum of the first and fifth digits equals 5?

(13) How many six-digit numbers are there in which all digits are of the same type (either all even or all odd)?

(14) How many positive four-digit numbers contain exactly one digit "1" and exactly one digit "3"?

(15) How many three-digit numbers have at least one even digit?

(16) How many four-digit numbers are there such that no two even digits are adjacent?

(17) How many positive integers less than 1000 do not contain the digit 7?

(18) How many positive five-digit numbers with distinct digits satisfy the condition that the positive difference between the first and last digit equals 2?

(19) When writing all the numbers from 1 to 100, how many times do we write the digit 6?

(20) How many palindromic numbers are there that have:

(a) 6 digits, formed from the digits $\{1,2,3,4,5,6,7,8\}$.

(b) 7 digits, formed from the same set of digits above.

(c) 5 digits, formed from the digits $\{0,1,2,3,4,5,6,7,8\}$.

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Second: Counting Words

We can calculate the number of English "words" made up of n letters using the Multiplication Principle directly (the "words" do not have to have any actual meaning).

Example: The number of words consisting of 4 letters is:

1st letter		2nd letter		3rd letter		4th letter	
Any letter		Any letter		Any letter		Any letter	**Number of Words**
26 ways	×	26 ways	×	26 ways	×	26 ways	$= 26^4$

Exercises:

(21) How many words can be formed from 5 different letters?

(22) In how many ways can the letters of the word PRODUCT be arranged?

(23) How many 6-letter words can be formed if the 1st, 3rd, and 6th letters are different?

(24) How many 3-letter words contain the letter M exactly once?

(25) How many 3-letter words contain the letter M at least once, if no other letter is repeated?

(26) How many 3-letter words contain the letter M at least once, if repetition of other letters is allowed (that is, another letter may appear twice)?

(27) The Saudi team for the Cybersecurity Olympiad invented their own language consisting of only four letters. In their language, the longest word contains at most 6 letters. How many words are in this language?

(28) We define a good word as a word made up only of the letters A, B, and C (it's allowed for any one of these letters not to appear) such that B does not immediately follow A, M does not immediately follow B, and A does not immediately follow C. How many 7-letter good words are there?

(29) How many palindromic words of length seven are there that do not begin with a vowel?

The vowels are (A, E, I, U, Y).

(30) In how many ways can we write all the English letters in a row so that there are exactly five letters between the two letters x and y?

Permutations

Permutation: is when we have a set of n distinct objects and we want to choose k of them, taking order into account and without repetition. The number of possible arrangements is given by:

$$^{n}P_{k} = n \times (n - 1) \times (n - 2) \times \ldots \times (n - k + 1)$$

and is read as "n permute k."

After learning about the concept of permutations, let's look at the following exercise:

Example:

In how many ways can the friends Ahmed, Mohammed, Younes, Hamza, and Hashem stand in a row for a group photo? In how many ways can 3 of them stand in a row? And in how many ways can at least 3 of them stand in a row?

Solution:

When arranging all five friends in a row: $^{5}P_{5} = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

When arranging only three of them: $^{5}P_{3} =$ **60**

When arranging at least three of them $= {}^{5}P_{3} + {}^{5}P_{4} + {}^{5}P_{5} = 60 + 120 + 120 =$ **300 ways**.

Note: We can write $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$ as $5!$, read as "five factorial."

Try calculating $4!$ and $3!$, what do you notice?

Note: For any positive integer n , We have $n! = (n - 1)! \cdot n$

Factorial: The factorial of n is the product of all positive integers less than or equal to n, and it is denoted by the exclamation mark (!):

$$n! = n \times (n-1) \times (n-2) \times \dots \times 3 \times 2 \times 1$$

Notice that:

$$^{n}P_{n} = n \times (n-1) \times (n-2) \times \dots \times 3 \times 2 \times 1$$

represents the number of ways to arrange n objects in a row, taking order into account.

Exercises:

(31) A football stadium has fifteen gates. In how many ways can you enter through one gate and exit through another?

(32) In a conference hall, there are 7 lights. To keep the lighting suitable, at least one light must be on. In how many ways can this be done?

Note:

Using the same reasoning as in the previous exercise, we can directly deduce from the multiplication principle that the number of subsets of a set containing n elements is: 2^n.

Since each element has two possibilities, it either belongs to the subset or does not. So:

Number of subsets: A set with n elements has 2^n subsets.

Exercises:

(33) Given the set $\{1, 2, 3, \ldots, 8\}$:

(a) How many subsets does it have?

(b) How many subsets contain no odd numbers?

(c) How many subsets contain exactly one odd number?

(34) A square board is divided into identical small squares, with dimensions 2026×2026. We want to shade squares on the board so that exactly one square is shaded in each row and in each column. (Assume that rotation and reflection give different patterns.)

(a) In how many ways can this be done?

(b) If shading in the corner squares is not allowed, in how many ways can it be done?

(35) How many four-digit numbers are there with exactly two identical digits, and whose thousands digit is less than 3?

From a set containing 9 elements, how many non-empty subsets contain an even number (36) of elements?

How many 5-digit numbers are there with distinct digits given even first and last digits(37)

In a football league, the organizers want to select 3 teams to be awarded at the annual (38) ceremony, with the selected teams receiving different ranks: first, second, and third. Each chosen team selects one player from its squad to receive the award. If the league has 12 teams and each team has 25 players, in how many ways can the three teams be selected along with their respective players?

Solutions

Algebra Solutions

Percent:

Exercises:

(1)

We have:

$$\frac{7}{x+y}=\frac{k}{x+z}\Longrightarrow k(x+y)=7(x+z)\longrightarrow(1)$$

In the same way

$$\frac{k}{x+z}=\frac{11}{z-y}\Longrightarrow k(z-y)=11(x+z)\longrightarrow(2)$$

By adding the two equations,

$$k(x+y)+k(z-y)=18(x+z)$$

$$\Rightarrow kx+ky+kz-Ky=18(x+z)$$

$$\Rightarrow k(x+z)=18(x+z)\Longrightarrow k=18$$

(2)

Since:

$$\frac{b+2}{2}=\frac{8}{c+3}\Longrightarrow(b+2)(c+3)=16$$

We have the following possible factor pairs:

- $(b+2)(c+3)=1\times 16\Longrightarrow b=-2\,,c=13$
  This is rejected because b is a natural number
- $(b+2)(c+3)=2\times 8\Longrightarrow b=0\,,c=5$
  This is rejected because b is a natural number
- $(b+2)(c+3)=4\times 4\Longrightarrow b=2\,,c=1$
  This case is possible
- $(b+2)(c+3)=8\times 2\Longrightarrow b=6\,,c=-1$
  This is rejected because c is a natural number
- $(b+2)(c+3)=16\times 1\Longrightarrow b=14\,,c=-2$
  This is rejected because c is a natural number

Thus, the only valid solution is $b=2\,,c=1\Longrightarrow a=5$

(3)

Let the number be x. Then

$$\frac{120}{100}x = 36 \Rightarrow x = 36 \times \frac{100}{120} = 30$$

(4)

$$\frac{2890 - 2023}{2890} \times 100 = \frac{867}{2890} \times 100 = 30$$

(5)

She gave her teacher 40%, leaving her with 60%

$$60 \times \frac{60}{100} = 36$$

She then gave her friend **one quarter** of what remained, meaning she kept **three quarters** of the remainder

$$36 \times \frac{3}{4} = 27$$

She then ate **one third** of what was left, so she kept **two thirds** of it

$$27 \times \frac{2}{3} = 18$$

(6)

The sum of the interior angles of a pentagon is:

$$(5 - 2) \times 180 = 540^\circ$$

And the sum of the ratio parts is

$$2 + 3 + 4 + 5 + 6 = 20$$

Thus, the value of one part is

$$\frac{540}{20} = 27^\circ$$

The largest angle corresponds to 6 parts, so

$$6 \times 27 = 162^\circ$$

(7)

$$b = \frac{105}{100}a \quad , \quad b = \frac{85}{100}c$$

$$\Longrightarrow \frac{105}{100}a = \frac{85}{100}c$$

$$\Longrightarrow \frac{a}{c} = \frac{85}{100} \times \frac{100}{105} = \frac{85}{105} = \frac{17}{21}$$

(8)

Let the original length be L and the original width be W.

Thus, the original area is $L \times W$

After the increase:

The new dimensions become:

$$\frac{150}{100}L \quad , \quad \frac{120}{100}W$$

The new area is

$$\frac{150}{100}L \times \frac{120}{100}W = \frac{180}{100}LW$$

Therefore, the percentage increase in area is

$$\frac{\frac{180}{100}LW - LW}{LW} \times 100 = 80\%$$

(9)

Every day, 20% is sold, leaving 80% of the fish remaining each day. Therefore:

After one day, $\frac{80}{100}$ of the quantity remains.

After two days, $\frac{80}{100} \times \frac{80}{100} = \frac{64}{100}$ of the initial amount

Because after two days (i.e., on Tuesday) **2,000 fish** remain, we set:

$$\frac{64}{100}x = 2000 \Longrightarrow x = 2000 \times \frac{100}{64} = 3125$$

(10)

Since:

$$t = \frac{u}{4} \Longrightarrow 4t = u$$

Therefore

$$\frac{4t}{2u} = \frac{u}{2u} = \frac{1}{2}$$

$$\Longrightarrow 4t : 2u = 1 : 2$$

(11)

Since:

$$\frac{x}{yz} : \frac{y}{zx} = 1 : k \Longrightarrow \frac{k}{1} = \frac{y}{zx} \times \frac{yz}{x} = \frac{y^2}{x^2}$$

And

$$yz : zx = 1 : 2 \Longrightarrow \frac{yz}{zx} = \frac{y}{x} = \frac{1}{2}$$

Therefore

$$k = \frac{y^2}{x^2} = \left(\frac{y}{x}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

One-Variable Linear Equations:

Exercises:

(1)

Since the given equation contains complex fractions in both sides,

it is better to simplify each side separately first. From

$$1-\frac{x-\frac{1+3x}{5}}{3}=1-\frac{5x-(1+3x)}{15}=\frac{15-2x+1}{15}=\frac{16-2x}{15}\quad \frac{x}{2}-\frac{2x-\frac{10-6x}{7}}{2}$$

$$=\frac{x}{2}-\frac{14x-(10-6x)}{14}=\frac{10-13x}{14}$$

it follows that

$$\frac{16-2x}{15}=\frac{10-13x}{14}$$

$$\Longrightarrow 14(16-2x)=15(10-13x)$$

$$\Longrightarrow 224-28x=150-195x$$

$$\Longrightarrow x=-\frac{74}{167}$$

(2)

By moving 3 in the given equation to the left hand side, it follows that

$$\left(\frac{x-a-b}{c}-1\right)+\left(\frac{x-b-c}{a}-1\right)+\left(\frac{x-c-a}{b}-1\right)=0$$

$$\Longrightarrow \frac{x-a-b-c}{c}+\frac{x-a-b-c}{a}+\frac{x-a-b-c}{b}=0$$

$$\Longrightarrow (x-a-b-c)\left(\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right)=0$$

$$\because \frac{1}{c}+\frac{1}{a}+\frac{1}{b}>0, \therefore x-a-b-c=0$$

$$\Longrightarrow x=a+b+c$$

(3)

$$(x-3)^2+(x+1)^2+(4x-5)^2=0$$

We observe that each term is a non-negative quantity, and their sum equals **zero**. For this to occur, each term must be **zero** at the same time. However, this requires:

$$x=3, x=-1, x=\frac{5}{4}$$

simultaneously, which is impossible. Therefore, there is **no real solution** to this equation.

(4)

Removing the denominator of the given equation yields

$$20(ax+b)-4(5x+2ab)=520ax+20b-20x-8ab=520(a-1)x=5-20b+8ab$$

$i)$ When $a\neq 1$:

$$x=\frac{5-20b+8ab}{20(a-1)}$$

$ii)$ When $a=1$ and $b=\frac{5}{12}$:

the equation becomes

$$0\cdot x=0$$

so, any real number is a solution for x.

(iii) When $a=1$ and $b\neq\frac{5}{12}$:

the equation becomes

$$0\cdot x=5-12b$$

so, no solution for x.

5)

Change the given equation to the form $(2a+3b-12)x=5-3a$ we have

$$2a+3b-12=0 \qquad and \qquad 5-3a=0$$

Therefore

$$a=\frac{5}{3} \qquad , \qquad b=\frac{12-2a}{3}=\frac{26}{9}$$

(6)

From the given equation

$$2a(x+6)=4x+1$$

$$\Longrightarrow (2a-4)x=1-12a$$

Since it has no solution, this implies

$$2a-4=0 \quad and \quad 1-12a\neq 0$$

$$\therefore a=2$$

(7)

From

$$x=\frac{12}{k}$$

which is a positive integer, k is also positive integer and k is a divisor of 12. So

$$k=1,2,3,4,6,12$$

The number of possible values of k is 6 .

(8)

$$\frac{1}{x}+\frac{1}{x+1}+\frac{1}{x+2}=\frac{13}{12}$$

$$\Longrightarrow \frac{3}{x+2}<\frac{13}{12}<\frac{3}{x}$$

$$\Longrightarrow \frac{x+2}{3}>\frac{12}{13}>\frac{x}{3}$$

$$\Longrightarrow x+2>\frac{36}{13}>x$$

$$\Longrightarrow 13(x+2)>36>13x$$

$$\Longrightarrow x<3$$

$$\therefore x=1\,,2$$

By checking, $x=1$ does not satisfy the original equation, and $x=2$ satisfies the given equation. Thus, $x=2$ the only positive value that satisfies the equation.

(9)

Substituting 4 into the given equation as x, it follows that

$$3a - 4 = \frac{4}{2} + 3 \Longrightarrow 3a = 9 \Longrightarrow a = 3$$

$$\therefore (-a)^2 - 2a = 9 - 6 = 3$$

(10)

$$\frac{n(x-n) - m(x-m)}{mn} = \frac{m}{n}$$

$$\Longrightarrow (n-m)x - n^2 + m^2 = m^2$$

$$\Longrightarrow (n-m)x = n^2$$

When $n \neq m$, we have

$$x = \frac{n^2}{n-m}$$

When $n = m$, no solution.

Multi-Variable Linear Equations:

Exercises:

(1)

$$\frac{a_1}{a_2} = \frac{1}{2} = \frac{b_1}{b_2} = \frac{5}{10} = \frac{c_1}{c_2} = \frac{3}{6}$$

The system has **infinitely many solutions**.

(2)

$$(a)\ x = 10\,, y = 3$$

$$(b)\ x = 4\,, y = 1$$

$$(c)\ x = 8\,, y = 6$$

$$(d)\ x = 5\,, y = 4$$

(3)

Let:

$$x + y = 42$$
$$x - y = 8$$

By adding the two equations:

$$2x = 50 \Rightarrow x = 25$$

By substituting in the first equation:

$$25 + y = 42 \Rightarrow y = 17$$

So the two numbers are 25 □ 17

(4)

We are told that $a = 6b$.

To convert our other information into an equation, we note that when it is b minutes after ، 3 o'clock, it is $b + 60$ minutes after 2 o'clock.

We are told that this time is 15 minutes later than when the time was a minutes after 2 o'clock.

So, we have:

$$\begin{cases} a = 6b \\ a + 15 = b + 60 \end{cases}$$

$\Longrightarrow b = 9$.

Therefore, it was $3:09$ when she looked at her watch for the second time.

(5)

$$\frac{x-y}{5}-\frac{x+y}{4}=\frac{1}{2}\longrightarrow(1)$$
$$2(x-y)-3(x+y)+1=0\longrightarrow(2)$$

Simplifying the first equation, we have

$$\frac{x-y}{5}-\frac{x+y}{4}=\frac{1}{2}\Longrightarrow 4(x-y)-5(x+y)=10\Longrightarrow x+9y=-10\longrightarrow(3)$$

Simplifying the second equation, we have

$$2(x-y)-3(x+y)+1=0\Longrightarrow x+5y=1\longrightarrow(4)$$

By $(3)-(4)$,

$$4y=-11\Longrightarrow y=-\frac{11}{4}$$

From (4),

$$x=1-5y=1+\frac{55}{4}=\frac{59}{4}$$

(6)

$$\begin{cases}5.4x+4.6y=104\longrightarrow(1)\\4.6x+5.4y=96\longrightarrow(2)\end{cases}$$

Notice the feature of coefficients, by $(1)+(2)$, we obtain

$$10x+10y=200\Longrightarrow x+y=20\longrightarrow(3)$$

By $(1)-(2)$, it follows that

$$0.8x-0.8y=8\Longrightarrow x-y=10\longrightarrow(4)$$

By $(3)+(4)$, we obtain

$$2x=30\Longrightarrow x=15$$

From (3),

$$x+y=20\Longrightarrow 15+y=20\Longrightarrow y=5$$

(7)

$$\begin{cases}x+(5x+y)=16\longrightarrow(1)\\ \quad 5x+y=7\longrightarrow(2)\end{cases}$$

By using 7 to substitute $5x+y$ in the first equation, we obtain

$$x+2(5x+y)=16\Longrightarrow x+2(7)=16\Longrightarrow x=2$$

Then from the second equation,

$$5x+y=7\Longrightarrow 5(2)+y=7\Longrightarrow y=-3$$

(8)

$$\begin{cases} \dfrac{x}{2} = \dfrac{y}{3} = \dfrac{z}{5} \longrightarrow (1) \\ x + 3y + 6z = 15 \longrightarrow (2) \end{cases}$$

Let,

$$t = \frac{x}{2} = \frac{y}{3} = \frac{z}{5} \Longrightarrow x = 2t \,, y = 3t \,, z = 5t$$

Substituting into the second equation, we have

$$x + 3y + 6z = 15 \Longrightarrow 2t + 9t + 30t = 15 \Longrightarrow t = \frac{15}{41}$$

Thus

$$x = \frac{30}{41} \,, y = \frac{45}{41} \,, z = \frac{75}{41}$$

(9)

$$\begin{cases} x + 2y = 5 \\ y + 2z = 8 \\ z + 2u = 11 \\ u + 2x = 6 \end{cases}$$

From the given equations we have the cyclic substitutions

$$x = 5 - 2y \,, \qquad y = 8 - 2z \,, \qquad z = 11 - 2u \,, \qquad u = 6 - 2x$$

By substituting them sequentially, we have

$$\begin{aligned} x = 5 - 2y = 5 - 2(8 - 2z) = -11 + 4z = -11 + 4(11 - 2u) = 33 - 8u \\ = 33 - 8(6 - 2x) = -15 + 16x \end{aligned}$$

Therefore

$$x = -15 + 16x \Longrightarrow x = 1 \Longrightarrow u = 4 \,, z = 3 \,, y = 2$$

(10)

$$\begin{cases} 5x - y + 3z = a \longrightarrow (1) \\ 5y - z + 3x = b \longrightarrow (2) \\ 5z - x + 3y = c \longrightarrow (3) \end{cases}$$

By $2 \times (1) + (2) - (3)$, it follows that

$$14x = 2a + b - c \Longrightarrow x = \frac{2a + b - c}{14}$$

By $2 \times (2) + (3) - (1)$, it follows that

$$14y = 2b + c - a \Longrightarrow y = \frac{2b + c - a}{14}$$

Similarly, by $2 \times (3) + (1) - (2)$, we have

$$14z = 2c + a - b \Longrightarrow z = \frac{2c + a - b}{14}$$

(11)

By substituting the solution $(2\,,1)$ into equations, we obtain

$$\begin{cases} 2a + b = 7 \longrightarrow (1) \\ 2b + c = 5 \longrightarrow (2) \end{cases}$$

After eliminating b, we obtain

$$4a - c = 9$$

(12)

$$\begin{cases} 3x - y = 5 \\ 2x + y - z = 0 \\ 4ax + 5by - z = -22 \end{cases} \quad , \quad \begin{cases} ax - by + z = 8 \\ x + y + 5 = c \\ 2x + 3y = -4 \end{cases}$$

From the first equation of the first system, we get $y = 3x - 5$

By substituting it into the last equation of the second system, it follows that

$$2x + 3(3x - 5) = -4$$

So $x = 1\,, y = -2$

Then, the second equation of the first system yields $z = 0$

Thus, from the second equation of the second system, $c = 4$

By solving the system

$$4a - 10b = -22, \qquad a + 2b = 8$$

the solution for a and b is obtained:

$$a = 2, b = 3.$$

Therefore, $a = 2\,, b = 3\,, c = 4$

(13)

$a)$ When

$$\frac{k}{6} \neq \frac{-1}{3}$$

i.e. $k \neq -2$ the system has unique solution

$$x = 0\,, y = -\frac{1}{3}$$

$b)$ When

$$\frac{k}{6} = -\frac{1}{3}$$

The system has infinitely many solutions.

$c)$ Thus, it's impossible that the system has no solution.

(14)

$$\begin{cases} 2020(x-y) + 2021(y-z) + 2022(z-x) = 0 \\ \\ 2020^2(x-y) + 2021^2(y-z) + 2022^2(z-x) = 2021 \end{cases}$$

Let $u = x - y\,, v = y - z\,, w = z - x$. Then $u\,, v, w$ satisfy the following system of equations

$$\begin{cases} u + v + w = 0 \longrightarrow (1) \\ 2020u + 2021v + 2022w = 0 \longrightarrow (2) \\ 2020^2u + 2021^2v + 2022^2w = 2021 \longrightarrow (3) \end{cases}$$

By $2021 \times (1) - (2)$, we obtain

$$u - w = 0 \Longrightarrow u = w$$

From (1) again, we have $v = -2w$

By substituting it into (3), we have

$(2020^2 - 2 \cdot 2021^2 + 2022^2)w = 2021$

$\Longrightarrow [(2022 + 2021) - (2021 + 2020)]w = 2021$

$\Longrightarrow 2w = 2021$

$\Longrightarrow z - y = -v = 2w = 2021$

(15)

$$\begin{cases} x - y - z = 5 \longrightarrow (1) \\ y - z - x = 1 \longrightarrow (2) \\ z - x - y = -15 \longrightarrow (3) \end{cases}$$

By $(1) + (2) + (3)$,

$$x + y + z = 9 \longrightarrow (4)$$

By $(2) + (3)$, it follows that

$$-2x = -14 \Longrightarrow x = 7$$

By $(2) + (4)$ we obtain

$$2y = 10 \Longrightarrow y = 5$$

Similarly, by $(3) + (4)$ we obtain

$$2z = -6 \Longrightarrow z = -3$$

Thus, the solution is $x = 7$, $y = 5$, $z = -3$

(16)

$$\begin{cases} x - y + z = 1 \longrightarrow (1) \\ y - z + u = 2 \longrightarrow (2) \\ z - u + v = 3 \longrightarrow (3) \\ u - v + x = 4 \longrightarrow (4) \\ v - x + y = 5 \longrightarrow (5) \end{cases}$$

By adding the equations, we obtain

$$x + y + z + u + v = 15 \longrightarrow (6)$$

By $(1) + (2)$, $(2) + (3)$, $(3) + (4)$, $(4) + (5)$, $(5) + (1)$ respectively, we obtain

$$\begin{cases} x + u = 3 \\ y + v = 5 \\ z + x = 7 \\ u + y = 9 \\ v + z = 6 \end{cases}$$

By substituting into (6), we have

$$x = 0, y = 6, z = 7, u = 3, v = -1$$

Challenge Problems:

(1)

By adding the two equations:

$$\frac{2}{a} = 20 \Longrightarrow \frac{1}{a} = 10 \Longrightarrow a = \frac{1}{10}$$

By substituting in the second equation:

$$10 + \frac{1}{b} = 14 \Longrightarrow \frac{1}{b} = 4 \Longrightarrow b = \frac{1}{4}$$

(2)

By adding the three equations, we obtain:

$$2(x + y + z) = 18 \Longrightarrow\ x + y + z = 9$$

Substituting into the first equation gives $z = 1$,

From the second, we get $x = 5$ **, and from the third**, we obtain $y = 3$.

(3)

For example, $\frac{1}{2}, \frac{1}{3}$

To find a general form Let the two fractions be $\frac{1}{a}, \frac{1}{b}$,

Then we have

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{a} \cdot \frac{1}{b} \Longrightarrow \frac{b-a}{ab} = \frac{1}{ab}$$

which gives:

$$b - a = 1 \Longrightarrow b = a + 1$$

Thus, the two fractions are, in general

$$\frac{1}{a}, \frac{1}{a+1}$$

(4)

Let the width and length of the shaded rectangle be x and $3y$, respectively, and let the length of the unshaded rectangle be l.

Since each rectangle has the same perimeter, we have:

$$2(l + y) = 2(x + 3y)$$

Thus,

$$l + y = x + 3y$$

and therefore,

$$l = 3y - 2x.$$

Since the original shape is a square, we also have:

$$x + l = 3y.$$

Substituting gives:

$$x + 3y - 2x = 3y$$

which implies:

$$y = 3x.$$

Hence, the area of the shaded rectangle is:

$$3xy.$$

And the area of the unshaded rectangle is:

$$ly = y(x + 2y) = y(x + 6x) = 7xy.$$

Therefore, the ratio of the area of the shaded rectangle to the unshaded one is:

$$3xy: 7xy = 3: 7.$$

(5)

Let the number of children be n, and let the total amount of money be k halalas.

We have

$$60n = k - 210$$

$$\Longrightarrow k = 60n + 210.$$

From another side,

$$\frac{k+20}{n} = 70$$

$$\Longrightarrow k = 70n - 20.$$

$$\therefore 60n + 210 = 70n - 20,$$

and therefore

$$n = 23,$$

which is the number of children.

(6)

$$3600 = 2^4 \times 3^2 \times 5^2 = 2^a \times 3^b \times 4^c \times 5^d$$

$$\Longrightarrow b = 2\,, d = 2\,,\ 2^4 = 2^a \times 4^c$$

$$\because\ 4^c = 2^{2c}$$

$$\therefore\ 2^4 = 2^a \times 2^{2c} \Longrightarrow 4 = a + 2c \Longrightarrow a = 4 - 2c$$

But we are also given that: $a + b + c + d = 7$

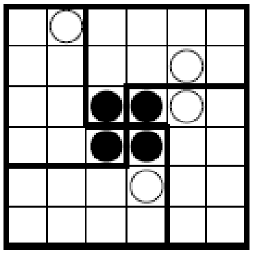
$$\Longrightarrow 4 - 2c + 2 + c + 2 = 7 \Longrightarrow c = 1$$

(7)

We want to have as many 5s as possible on the left. We will start by removing 1234 and leaving 5. Then we repeat the process: remove 1234 and leave 5. Unfortunately, we cannot make another 5 on the left. We are only allowed to remove the next 12. Therefore, the largest

number we can obtain is: 553451234512345.

(8)



(9)

The only child who could be in kindergarten is the 5-year-old, and because one of the children in kindergarten is a girl, the youngest child is a girl. Muhammad is not the youngest, and because he is younger than Najah, he is not the oldest either. Therefore, Muhammad is either **8 or 13** years old.

- Assuming Muhammad is 8 years old, then Raja is 13 or 15, and the possible pairs representing the ages of Nour and Raja are:

$$(5,13), (5,15), (13,15)$$

The only pair that satisfies the condition that the sum of their ages is divisible by 3 is when Nour is 5 years old and Raja is 13 years old.

- Assuming that Muhammad is 13 years old, then Raja is 15 years old, and the possible pairs representing the ages of Nour and Raja are:

$$(5,15), (8,15)$$

There is no possibility that the sum of the two ages can be divided by three.

Therefore, the conditions of the problem are satisfied **only when Nour is five years old**, which means that Nour must be a **girl**.

(10)

If everyone on the island always told the truth, we would have exactly 100 "yes" answers. Each liar gives **three** "yes" answers instead of just one, which increases the total number of "yes" answers by 2 for each liar.

Since the total number of "yes" answers given was:

$25 + 25 + 45 + 55 = 150,$

there are 50 **extra** "yes" answers.

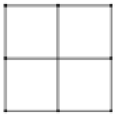
Therefore, the number of liars is:

$50 \div 2 = 25.$

(11)

The question is not easy to solve, and the goal is to encourage the student's creativity

If we look at any 2×2 square divided into 4 small squares inside the table.



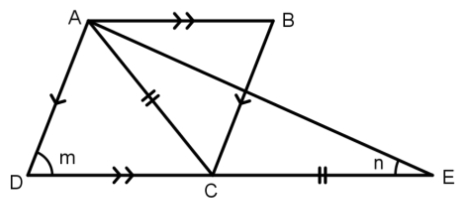
Let's call it a "block." We will find that any two small squares in it will share a corner or side. Therefore, we cannot color more than one small square black inside any block in the table. Now we ask the question: What is the largest number of separate (i.e., non-overlapping) blocks into which we can divide the given table? The answer is 16 blocks!

Therefore, the largest number of small squares that can be colored black under the conditions of the question is 16, and we cannot color 17 small squares in this way.

Geometry solutions

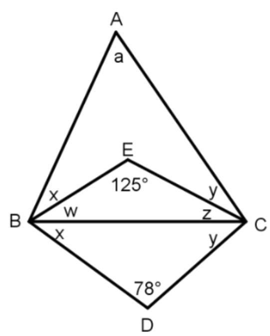
Solutions of Revision Exercises:

(1)



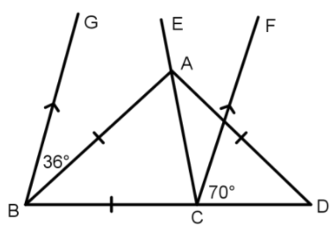
$$CA = CE \Longrightarrow \angle EAC = \angle AEC = n \Longrightarrow \angle ACD = n + n = 2nAD = CD \Longrightarrow \angle DAC = \angle DCA = 2nD + \angle DAC + \angle DCA = 180 \Longrightarrow 4n + m = 180$$

(2)



$$x + y = 180 - 78 = 102w + z = 180 - 125 = 55A = 180 - (x + y + w + z) = 180 - (102 + 55) = 23°$$

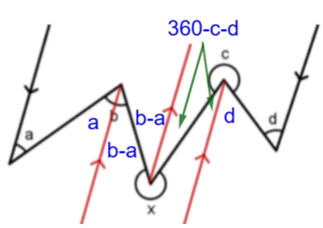
(3)



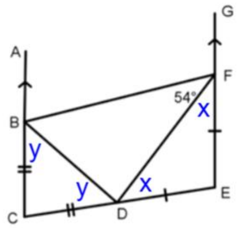
$$\angle DCF = \angle CBG = 70 \Longrightarrow \angle ABC = 70 - 36 = 34° \Longrightarrow \angle ADC = \angle ABC = 34, \angle BAC = \angle BCA = \frac{180-34}{2} = 73°\angle BAD = 180 - (34 + 34) = 112° \Longrightarrow \angle DAC = 112 - 73 = 39 \Longrightarrow \angle EAD = 180 - 39 = 141°$$

(4)

$$x + b - a + 360 - c - d = 360 \Longrightarrow x = c + d + a - b$$

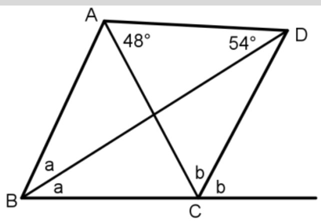


(5)



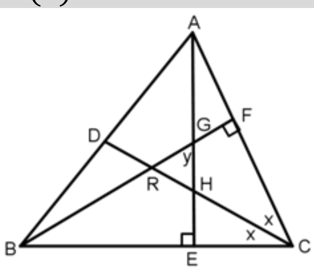
$2x + \angle FED = 180\ 2y + \angle BCD = 180 \quad \angle FED + \angle BCD$
$= 180 \Longrightarrow 2x + 2y = 180 \Longrightarrow x + y = 90$
$\Longrightarrow \angle BDF = 180 - (x + y) = 90$

(6)



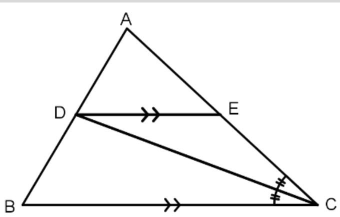
$\angle BAC = 2b - 2a\angle BDC = b - a2b = a + 78 \quad , \quad a = \frac{2}{3}b$
$\Longrightarrow 2b = \frac{2}{3}b + 78 \Longrightarrow b = 58.5° \ , = a = 39°$

(7)



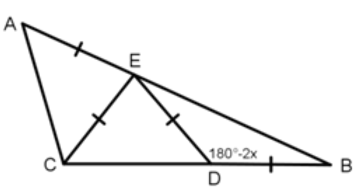
$\angle FRC = 90 - x\angle GHR = \angle EHC$
$= 90 - x\Delta GHR\colon y + (90 - x) + (90 - x) = 180$
$\Longrightarrow y = 2x$

(8)



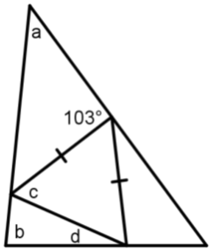
$\angle ADE = \angle B = 70°,$
$\angle EDC = \angle BCD = \frac{40}{2} = 20°$
$\Longrightarrow \angle BDC = 180 - (70 + 20) = 90°$

(9)



$\angle DBE = \angle DEB = x\angle DCE = \angle CDE = 2x\angle AEC$
$= \angle DCE + \angle DBE = 3x\angle ACE = \angle CAE$
$= \frac{180 - 3x}{2}$

(10)



$\frac{3}{2}a = c \quad , 2a = b \quad , \frac{1}{3}a = d77 - a + c = b + d \Longrightarrow 77 - a + \frac{3}{2}a = 2a + \frac{1}{3}a \Longrightarrow 77 \times 6 - 6a + 9a = 12a + 2a \Longrightarrow 77 \times 6 = 11a \quad \Longrightarrow a = 42°$

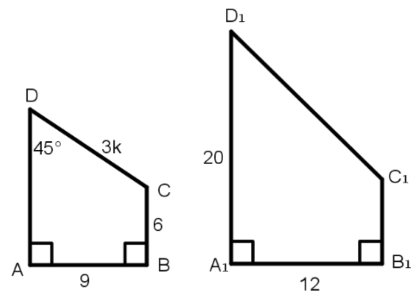
$b = \frac{42}{2} = 21 \quad , \quad c = \frac{3 \times 42}{2} = 63 \quad , \quad d = \frac{42}{3} = 14$

Solutions of Similarity Exercises:

$(1-8)$

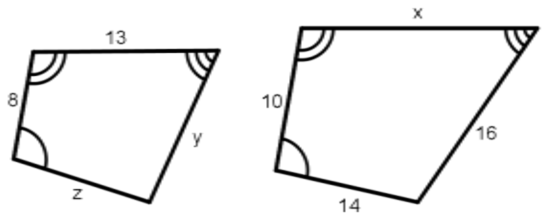
Exercise	Polygons	Cannot be similar	Can be similar	Always similar
(1)	**Equilateral triangles**			√
(2)	**Isosceles triangles**		√	
(3)	**Squares**			√
(4)	**Rhombuses**		√	
(5)	**Right triangles**		√	
(6)	**Scalene Triangles**		√	
(7)	**Rectangles**		√	
(8)	**Right triangle and acute triangle**	√		

$(9-16)$



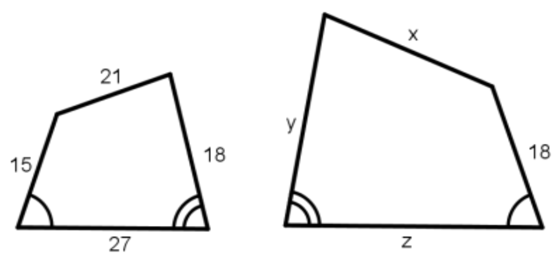
(9)	Similarity ratio of $ABCD$ ، $A_1B_1C_1D_1$.	$\frac{3}{4}$
(10)	Type of quadrilateral $A_1B_1C_1D_1$	Trapezoid
(11)	Measure of $m\angle D_1$	$45°$
(12)	Measure of $m\angle C_1$	$180-45=135°$
(13)	Measure of C_1B_1	$\frac{12}{9}=\frac{C_1B_1}{6} \Longrightarrow C_1B_1=8$
(14)	Measure of AD	$\frac{12}{9}=\frac{20}{AD} \Longrightarrow AD=15$
(15)	Measure of C_1D_1	$\frac{12}{9}=\frac{C_1D_1}{3k} \Longrightarrow C_1D_1=4k$
(16)	Ratio of the perimeters of the quadrilaterals	$\frac{3}{4}$

(17)



$$\frac{10}{8}=\frac{x}{13}\Longrightarrow x=\frac{65}{4}\frac{10}{8}=\frac{16}{y}\Longrightarrow y=\frac{64}{5}\frac{10}{8}=\frac{14}{z}\Longrightarrow z=\frac{56}{5}$$

(18)



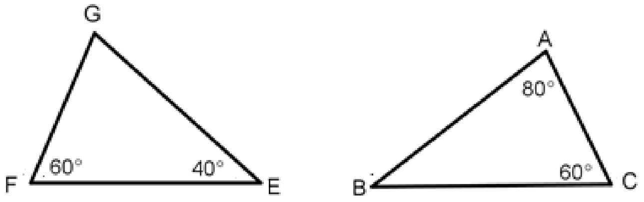
$$\frac{18}{15}=\frac{y}{18}\Longrightarrow y=\frac{108}{5}\frac{18}{15}=\frac{z}{27}\Longrightarrow y=\frac{162}{5}\frac{18}{15}=\frac{x}{21}\Longrightarrow x=\frac{126}{5}$$

Solutions of Triangle similarity exercises:

$(1 - 9)$

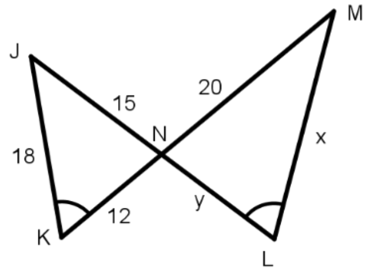
Exercise	Answer
1	Similar
2	Not Similar
3	Similar
4	Similar
5	Similar
6	Inconclusive
7	Similar
8	Similar
9	Similar

(10)



$$\angle ABC = 40°$$
$$\Delta ABC \sim \Delta GEF$$
$$\frac{AB}{GE} = \frac{BC}{EF} = \frac{AC}{GF}$$

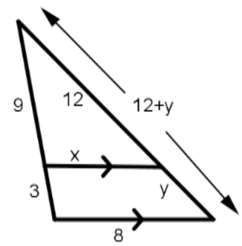
(11)



$$\Delta JKN \sim \Delta MLN$$

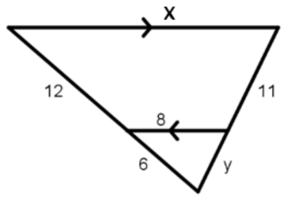
$$\frac{15}{20} = \frac{12}{y} \Longrightarrow y = 16 \quad \frac{15}{20} = \frac{18}{x} \Longrightarrow x = 24$$

(12)



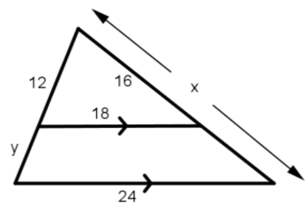
$$\frac{9}{12} = \frac{12}{12+y} \Longrightarrow y = 3 \quad \frac{9}{12} = \frac{x}{8} \Longrightarrow x = 6$$

(13)



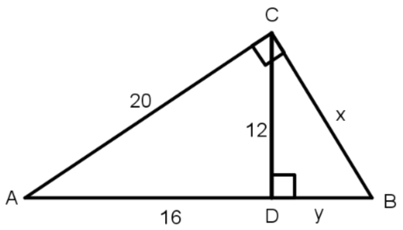
$$\frac{6}{18} = \frac{y}{11+y} \Longrightarrow y = \frac{11}{2} \quad \frac{6}{18} = \frac{8}{x} \Longrightarrow x = 24$$

(14)



$$\frac{18}{24} = \frac{12}{12+y} \Longrightarrow y = 4 \quad \frac{18}{24} = \frac{16}{x} \Longrightarrow x = \frac{64}{3}$$

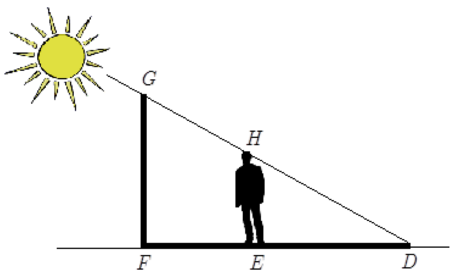
(15)



a) $\Delta ABC \sim \Delta ACD \sim \Delta CBD$

b) $\frac{BD}{CD} = \frac{CD}{AD} = \frac{CB}{AC} \Longrightarrow \frac{y}{12} = \frac{12}{16} = \frac{x}{20} \Longrightarrow x = 15 \ , \ y = 9$

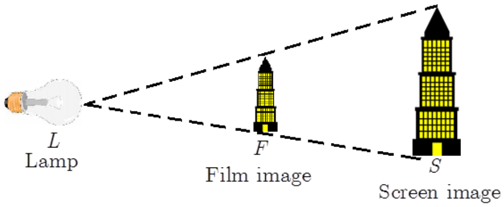
(16)



$$\Delta HDE \sim GDF$$

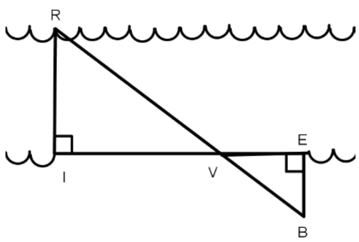
$$\frac{HE}{GF} = \frac{DE}{FD} \Longrightarrow \frac{2}{GF} = \frac{1.6}{4.4} \Longrightarrow GF = 5.5m$$

(17)



$$\frac{6\ cm}{24\ m} = \frac{x\ cm}{2.2\ m} \Longrightarrow x = 0.55\ cm$$

(18)



$$\Delta VEB \sim \Delta VIR$$

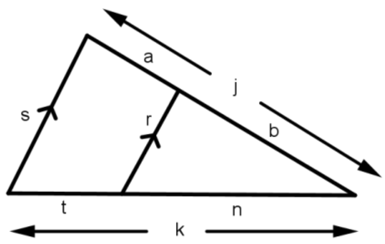
$$\Rightarrow \frac{20}{63} = \frac{15}{IR} \Rightarrow IR = 27.25\ m$$

(19 – 24)

Exercise	Similar Triangles	Theorem
(19)	$\Delta ABC \sim \Delta GNK$	$\boldsymbol{SAS}$
(20)	$\Delta ABC \sim \Delta ARS$	$\boldsymbol{SAS}$
(21)	$\Delta ABC \sim \Delta THJ$	$\boldsymbol{AA}$
(22)	$\Delta ABC \sim \Delta MBL$	$\boldsymbol{SAS}$
(23)	$\Delta ABC \sim \Delta XRN$	$\boldsymbol{SSS}$
(24)	$\Delta ABC \sim \Delta AEF$	$\boldsymbol{AA}$

Solutions of Proportional Lengths:

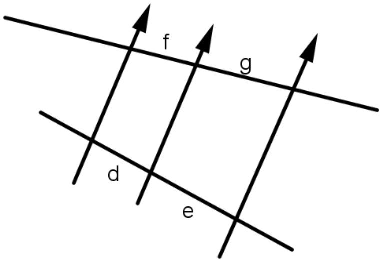
(1)



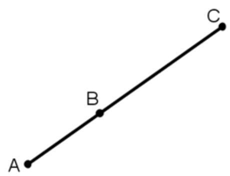
Part	Relation	Answer
(a)	$\frac{r}{s}=\frac{a}{b}$	X
(b)	$\frac{t}{k}=\frac{a}{j}$	√
(c)	$\frac{j}{a}=\frac{s}{r}$	√
(d)	$\frac{r}{s}=\frac{n}{k}$	√
(e)	$\frac{a}{b}=\frac{n}{t}$	X
(f)	$\frac{b}{j}=\frac{t}{k}$	X

(2)

Part	Relation	Answer
(a)	$\frac{d}{f}=\frac{g}{e}$	X
(b)	$\frac{f}{g}=\frac{e}{d}$	X
(c)	$\frac{g}{f}=\frac{e}{d}$	√
(d)	$\frac{d}{f}=\frac{e}{g}$	√

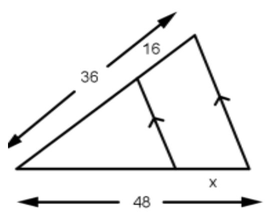


(3 – 6)



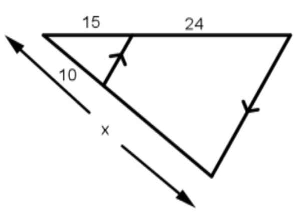
(6)	(5)	(4)	(3)	
37.5	21	15	6	AB
62.5	35	25	10	BC
100	56	40	16	AC

(7)



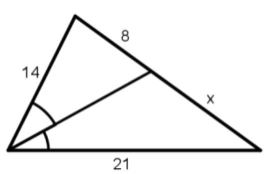
$$\frac{x}{48} = \frac{16}{36} \Longrightarrow x = \frac{64}{3}$$

(8)



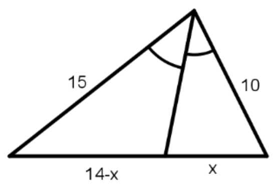
$$\frac{10}{x} = \frac{15}{39} \Longrightarrow x = 26$$

(9)



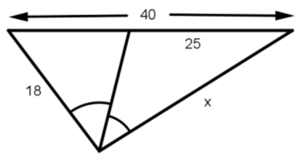
$$\frac{14}{21} = \frac{8}{x} \Longrightarrow x = 12$$

(10)



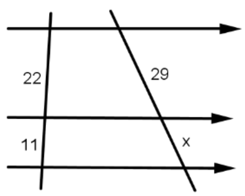
$$\frac{15}{10} = \frac{14-x}{x} \Longrightarrow x = \frac{28}{5}$$

(11)



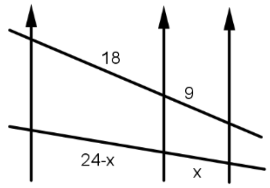
$$\frac{15}{25} = \frac{18}{x} \Longrightarrow x = 30$$

(12)



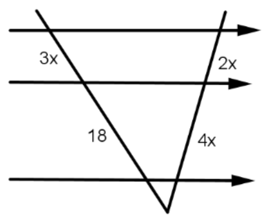
$$\frac{22}{11} = \frac{29}{x} \Longrightarrow x = 14.5$$

(13)



$$\frac{9}{18} = \frac{x}{24-x} \Longrightarrow x = 8$$

(14)



$$\frac{18}{3x} = \frac{4x}{2x} \Longrightarrow x = 3$$

Solutions of Even and Odd Numbers:

(1)

We know that the number is even if it is divisible by 4 (since it will be divisible by 2 after dividing it by 2 once. But we know from the divisibility rule of 4, the last 2 digits have to be divisible by 4. But 38 is not divisible by 4. Thus, the result would be odd.

(2)

We cannot divide the books, since the sum of 5 odd numbers is odd while 50 is even.

(3)

Since a, b are two consecutive integers. One of them is even while the other is odd. But their multiplication is c. which means that c is even. So the numbers a, b, c have two evens and one odd. And their squares should have the same parity (square of even is even and square of odd is odd). Thus, $N = even + even + odd = odd$.

(4)

The pattern started with 1,2, if we continue the pattern we will notice that the parity of the numbers follows the pattern $(O, E, O, O, E, O, O, E, O, \dots)$. To decide the parity of the 2003^{th} term, we divide 2003 by 3 to get a remainder of 2. Thus, the numbers will be divided as O, E, O many times and then we will be left with O, E at the end (since the remainder is 2). So, the 2003^{th} will be even.

(5)

Let $k = (x + 1)(y + 2)(z + 3)$, then we have three cases:

Case 1: x, y are odd, then $x + 1$ is even. Thus, k will be even.

Case 2: x, z are odd, then $x + 1$ is even. Thus, k will be even.

Case 1: y, z are odd, then $z + 3$ is even. Thus, k will be even.

(6)

It is clear that the number of odd numbers in the list is odd (exactly 1009 odd numbers). So, if we add all the numbers the result would be odd. But since addition and subtraction does not affect the parity (only thing that affects the parity is the number of odd numbers). The final result must be odd.

(7)

(a) Yes, for example $1 + 2 - 3 + 4 + 5 + 6 - 7 - 8$.
(b) Not possible, since the sum $1 + 2 + \cdots + 9 = 45$ is odd. And as we mentioned before that subtraction or addition does not affect the parity. It means that the final result would be odd while 0 is even.

(8)

Not possible, since we have an addition/subtraction of 20 odd numbers (the number of odd numbers is even). Thus, the final score can only be even, but 39 is odd. So, it is not possible.

(9)

(a) It is possible, we put the oranges in the basket in the following way $1,2,1,2, \ldots ,1,2$.
(b) Not possible, this is clear by looking at the parities of the oranges in consecutive baskets. They have to be:

$$O, E, O, E, \ldots$$

But that means the first and last basket will have the same parity and so they cannot differ by 1.

(10)

(a), (b) Not possible, the worm can only jump an even integer number of steps. (c) is possible, that is by jumping $6\ cm$ twice to the right, followed by $8\ cm$ once to the left.

(11)

This is not an easy question; it seems possible but after many tries it would look impossible. But how do we prove that it is indeed impossible?

One way to tackle this problem is to look at the position of each of the numbers on the line (first number has position 1 while the last number has position 10). Then we can construct the following equations:

First, the sum of positions is $1 + 2 + \cdots + 10 = 55$. But looking at it from another point of view.

The sum of positions of 1,1 is $a + (a + 2) = 2a + 2$. (If we assume that a is the position of the first 1)

The sum of positions of 2,2 is $b + (b + 3) = 2b + 3$.

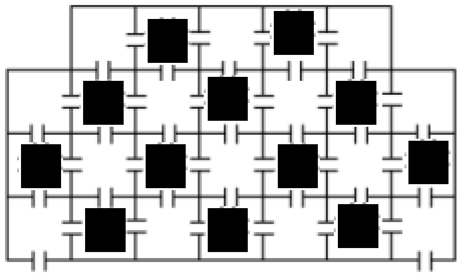
The sum of positions of 3,3 is $c + (c + 4) = 2c + 4$.

The sum of positions of 4,4 is $d + (d + 5) = 2d + 5$.

The sum of positions of 5,5 is $e + (e + 26) = 2e + 6$.

Note that the sum of the positions above is EVEN (since we only have two odd). However, 55 is odd. This shows that it is impossible to arrange the numbers as required by the problem statement.

(12)



This is a similar question to the previous one. It seems possible but after many tries we can see that it hard to show a way to do it. So, how can we show that it is impossible?

Notice that if we color the rooms as in the figure above. It means that we go from black to whit rooms and vice versa. We start at a white room and want to end at another white room passing through all the rooms. This means that the number of black and white rooms should be equal. But this is not the case. So it is impossible!

This question may seem unrelated to the topic. However, if you look closely, we are pairing the white and black rooms together (so it is not that the parity of black and white rooms has to be equal, but that **black** room has to be paired with a **white** room, and that is impossible as shown above)

Solutions for Prime and Composite Numbers:

(1)

Answer is (b).

Notice that x, y sum up to an odd number (75). Thus, one of them is even and another is odd. But since they are both primes, it means one of them has to be 2 while the other is 73. And this gives the solutions x, y= (2,73), (73,2).

(2)

First, we can rule out the prime numbers with one digit equal to 2,4,5,6,8. After that, a simple search among the primes with digits in 1,3,7,9 Gives the numbers 11,13,17,31,37,71,73,79,97.

(3)

We have $a(b^b + c) = 2000 = 2^4 \cdot 5^3$. Thus, $a = 2 \text{ or } 5$.

Trying $a = 5$, we find that $b^b c = 399 = 3 \cdot 7 \cdot 19$ And this has no solutions.

Trying $a = 2$, we find that $b^b c = 3^3 \cdot 11$ Which has solutions only if $b = 3, c = 37$.

Thus, $a + b + c = 2 + 3 + 37 = 42$.

(4)

Since $p + q$ is even, then we have two cases:

Case 1: p, q are both even. Thus, $p = q = 2$ and this means that $p - q = 0$ which is not a prime. So this case has no solutions.

Case:2 p, q are both odd. This means that $p - q = 2$ and then we have $p = q + 2$. Plugging this into the equation we get that:

$$(1+\frac{1}{2})^p(1-\frac{1}{3})^q = (\frac{3}{2})^{q+2}.(\frac{2}{3})^q = (\frac{3}{2})^{q+2}.(\frac{3}{2})^{-q} = (\frac{3}{2})^2 = \frac{9}{4}$$

(5)

Notice that any positive integer can be written in one of the three forms $3m, 3m+1, 3m+2$ such that $m \geq 0$. Now, we have three cases:

Case 1: n is of the form $3m$. Then $n+3=3m+3=3(m+1)$ which is a multiple of 3.

Case 2: n is of the form $3m+2$ Then $n+7=3m+9=3(m+3)$ which is a multiple of 3.

Case3: n is of the form $3m+1$. Then $n+7=3m+8, n+3=3m+4$ which And these two numbers are possibly primes.

Indeed, if you plug $n=4$ then $n+3=7, n+7=11$. Thus, the remainder of n when divided by 3 is 1.

(6)

When $p \geq 5$, then we can write p in the form $6k \pm 1$, such that $k \geq 1$ integer.

If $p=6k+1$ then $2p+1=12k+3=3(4k+1)$ and this is a composite number.

On the other hand, if $p=6k-1$ then $2p+1$ could be a prime number.

And $4p+1=24k-3=3(8k-1)$ is a composite number since it is a multiple of 3.

(7)

Since $\frac{a}{b}=\frac{65}{56}$. Then we can assume that $a=65t, b=56t$ such that t is a positive integer. Then we have $a+b=65t+56t=121t=11^2t$ which is a composite number.

(8)

We can rewrite the equation in the form: $(m-n)(m+n)-2(m+n)=19 \Rightarrow$ $(m+n)(m-n-2)=19$. It is clear that $(m+n)>(m-n-2)$. Thus, we have:

$$m+n=19, m-n-2=1$$

Which gives solutions $m, n=11,8$.

(9)

Notice that 1661, 1986 are relatively prime numbers (have no common factors. Moreover, the prime factorization of $1986=2\times 3\times 331$, we can notice that

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{331}=\frac{1661}{1986}$$

Thus, $p+q+r=2+3+331=336$.

(10)

Since p is the smallest possible integer such that $p = m + n + mn$, this means that m, n are the two smallest prime numbers which are 2,3. This means that $p = 2 + 3 + 2 \times 3 = 11$.

Therefore,

$$\frac{m^2 + n^2}{p^2} = \frac{13}{121}$$

(11)

Since $q = mn$ is a prime number. This means that one of m, n is 1 while the other is q. Thus, $p = q + 1$ are two consecutive prime integers. This can only be satisfied if $p = 2, q = 3$. This means that m, n are equal to 1 and 2.

(12)

Notice that if the sum of two integers is 129, this means that one of them has to be even. Therefore, one of p, q has to be 2 since they are prime numbers. Now, we have two cases:

Case 1: $p = 2$ means that $q = 17$ and therefore $p + q = 19$.

Case 2: $q = 2$ means that $p = 23$ and therefore $p + q = 25$.

This means that $p + q = \{19,25\}$.

(13)

It is clear that $p = 2,3$ does not satisfy the question statement. If $p = 5$ then 5,7,11,13,19 are all primes. Now, we have 5 cases:

Case 1: $p = 5k \neq 5$. Then p itself is not a prime since it will be divisible by 5.

Case 2: $p = 5k + 1$. Then $p + 14 = 5k + 15$ is not a prime since it will be divisible by 5.

Case 3: $p = 5k + 2$. Then $p + 8 = 5k + 10$ is not a prime since it will be divisible by 5.

Case 4: $p = 5k + 3$. Then $p + 2 = 5k + 5$ is not a prime since it will be divisible by 5.

Case 5: $p = 5k + 4$. Then $p + 6 = 5k + 10$ is not a prime since it will be divisible by 5.

Thus, the only solution is $p = 5$.

(14)

The only two consecutive prime integers are 2,3. Thus, $(n-1)+(3n-19)+(38-5n)+(7n-45)=6n-27$. Since p, q are 2,3 then $p+q+2p+2q=15$. And from $6n-27=15$ we get $n=7$ and we just need to check that it is correct.

(15)

Notice that we can pick $n=41k$. This makes $(41k)^2+41k+41=41(41k^2+k+1)$. And this number is composite for any k. Thus, we are done.

Solutions (**Combintorics**)

The Two Basic Counting Principles:

(1)

$2 \times 3 = 6$

(2)

$6 \times 7 = 42$

(3)

Without any ties at all, or with exactly two of them tied (either for first place or second place), or with all three tied.

$6 + 6 + 1 = 13$

(4)

1 , 3 , 5 , 2 — 1 , 4 , 2 , 5 – 1 , 5 , 2 , 4

3 passwords starting with 1

----

2 , 4 , 1 , 3 — 2 , 4 , 1 , 5 — 2 , 5 , 1 , 3 — 2 , 5 , 1 , 4

2 , 5 , 3 , 1 — 5 passwords starting with 2

----

3 , 1 , 4 , 2 — 3 , 1 , 5 , 2 —3 , 5 , 1 , 4 —

3 , 5 , 2 , 4 — 4 passwords starting with 3

----

4 , 1 , 3 , 5 — 4 , 1 , 5 , 3 — 4 , 1 , 5 , 2 —

4 , 2 , 5 , 1 — 4 , 2 , 5 , 1 — 5 passwords start with 4

----

5 , 1 , 4 , 2 — 5 , 2 , 4 , 1 — 5 , 3 , 1, 4 —

3 passwords starting with 5

Total: 3 + 5 + 4 + 5 + 3 = 20

(5)

We arrange the three subject groups in 6 ways, then arrange the math books ($3 \times 2 \times 1 = 6$ ways), the chemistry books ($4 \times 3 \times 2 \times 1 = 24$ ways), and the physics books ($5 \times 4 \times 3 \times 2 \times 1 = 120$ ways). Therefore, the total number of ways = $3 \times 2 \times 1 \times 6 \times 24 \times 120 = 103{,}680$ ways.

Number of Numbers and Strings:

(6)

$9 \times 9 \times 8 \times 7 = 4536$

(7)

$9 \times 10 \times 10 \times 10 \times 5 = 45000$

(8)

$(4 \times 4) + (3 \times 4) = 16 + 12 = 28$

(9)

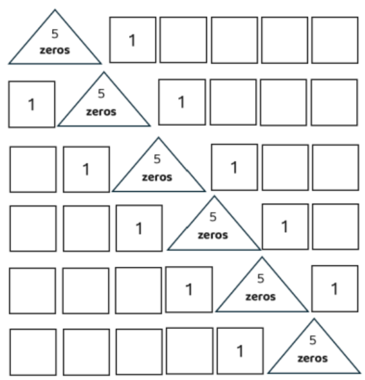
$3 \times 9 \times 9 = 243$

(10)

Divisible by 3: (1,8,3), (1,8,9), (7,8,9), (7,8,3), and each has six arrangements: $4 \times 6 = 24$

(11)

The five consecutive zeros can start in any position from 1 to 6, with the condition that they are not adjacent to any other zero.



If they are at the edge (starting at 1 or 6), then there are 4 free slots: 2^4 for each.

If they are in the middle (starting at 2–5), then there are 3 free slots: 2^3 for each case.

Total: $2^3 \cdot 4 + 2^4 \cdot 2 = 64$.

(12)

$5 \times 10^3 = 5000$

(13)

$(4 \times 5^5) + (5^6) = 28125$

(14)

We choose the position of "1" (4 ways) and "3" (3 ways), and the remaining digits have 8 possibilities each, then subtract those that start with zero: $4 \times 3 \times 8 \times 8 - 3 \times 2 \times 8 = 720$.

(15)

$$900 - 5^3 = 775$$

(16)

We forbid two even digits from being adjacent, and the first digit cannot be zero. The number of choices for odd digits is always *5*, and for even digits, it's *4* in the first (leftmost) position and *5* in the others. Allowed patterns: OOOO, OEOO, OOEO, OOOE, OEOE (each has 5^4) and EOOO, EOEO, EOOE (each has 4×5^3).

So: $5 \times 625 + 3 \times (4 \times 125) = 3125 + 1500 = 4625$ numbers.

(17)

$$9 \times 9 \times 9 = 729$$

(19)

In the ones place *10*, and in the tens place *10* → total $= 20$.

(20)

(a) $8^3 = 512$
(b) $8^4 = 4096$
(c) $9^2 \times 8 = 648$

.

(18)

We choose the ordered pair (first, last) such that |first − last| = 2, with zero not allowed in the first position. For first digits from 1 to 9, the number of valid pairs is 15 (1 gives 1 pair, 2 gives 2, 3–7 each give 2, 8 gives 1, 9 gives 1). After fixing the first and last digits, the three middle digits are chosen from the remaining eight distinct digits, in order: $8 \times 7 \times 6 = 336$. Therefore, the total number $= 15 \times 336 = 5040$.

Number of Words:

(21)

$26 \times 25 \times 24 \times 23 \times 22 = 7{,}893{,}600$

(22)

$7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

(23)

$(26 \times 25 \times 24) \times 26^3 = 274{,}185{,}600$

(24)

$3 \times 25 \times 25 = 1875$

(25)

Either M appears once $(3 \times 25 \times 24)$, twice (3×25), or three times (1). Adding them up: $1800 + 75 + 1 = 1876$.

(26)

$26^3 - 25^3 = 1951$

(27)

$$4^1 + 4^2 + 4^3 + 4^4 + 4^5 + 4^6 = 5460$$

(28)

We start with one letter from A, B, or C (3 ways). Each letter can be followed by only two allowed letters (A or C after A, A or B after B, 9 or C after C). Thus, the number of words doubles at each step. After 7 letters,

the total $= 3 \times 2^6 = 192$ good words.

(29)

$$21 \times 26^3 = 369{,}096$$

(30)

If there are five letters between x and y, the distance between them is 7 positions.

So, the pair can start in any of the positions 1 to 20 ($26 - 7 + 1 = 20$).

For each position, there are 2 possible orders (x before y or y before x).

Then, the remaining 24 letters can be arranged in any order: $24!$

Therefore, the total number

$$20 \times 2 \times 24 \times 23 \times 22 \ldots \times 2 \times 1$$

Permutations:

(31)

$$15 \times 14 = 210$$

(32)

All subsets except the empty set (when all the lights are off).

$$2^7 - 1 = 127$$

(33)

$$2^8 = 256$$

$$2^4 = 16$$

$$4 \times 2^4 = 64$$

(34)

(a) $2026!$

(b) We color the first row in 2024 ways (excluding the corner cells), and the last row in 2023 ways (excluding the corners and the cell chosen in the first row). For the remaining rows, there are 2024! ways. So Answer:

$$2024 \times 2023 \times 2024!$$

(35)

The thousands place can be 1 or 2, so there are 2 ways.

Case 1: If the thousands digit is the repeated digit, choose the repeated digit (2 ways), choose the second position for the repeated digit (3 ways), choose two different digits from the remaining digits ($9 \times 8 \div 2 = 36$ ways) and arrange them ($\times\ 2$).

Total $= 2 \times 3 \times 36 \times 2 = 432$ ways.

Case 2: If the thousands digit is not the repeated digit, choose the thousands digit (2 ways), choose the repeated digit from the remaining 9 digits (9 ways), choose the two positions for it among the remaining three places (3 ways), and choose one more different digit (8 ways).

Total $= 2 \times 9 \times 3 \times 8 = 432$ ways.

Adding both cases: $432 + 432 = 864$ numbers.

(36)

The first digit must be even and not zero → 4 choices (2, 4, 6, 8)

The last digit must be even and different from the first (0 is allowed) → 4 choices

The three middle digits are chosen from the remaining 8 digits and ordered:

$^{8}P_3 = 8 \times 7 \times 6 = 336$

Therefore, the total number is:

$336 \times 4 \times 4 = 5376$

(37)

The number of subsets of a set with 9 elements is: $2^9 = 512$

Since every subset of size an even number has a complement with size an odd number, the number of even subsets equals the number of odd subsets.

So each is: $\frac{512}{2} = 256$

But the empty set is included among the even subsets, and we want only the non-empty ones, so: $256 - 1 = 255$

(38)

Choosing 3 teams arranged in ranked positions: $^{12}P_3 = 12 \times 11 \times 10 = 1320$ because the order matters.

Choosing one player from each team: 25^3 since each team has 25 players.

Total number of ways: 1320×25^3