نسمو
NSMO

موهبة
Mawhiba

وزارة التعليم
Ministry of Education

National Science and Mathematics Olympiad NSMO

Mathematics 1

General Administrations Competition

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Introduction

Our exceptional sons and daughters,

We are delighted to congratulate you on successfully completing the **Cities and Governorates stage** and qualifying for the **General Administrations stage**, a significant and advanced step on your path toward mathematical challenge and innovation.

This **resource packet** is designed to **expand your understanding** across the four main branches of mathematics: **Combinatorics, Geometry, Algebra, and Number Theory**. We will focus on advanced concepts in **counting, triangle congruence, linear equations, and even and odd numbers.**

This stage aims to **refine your skills in analytical thinking** and help you **connect mathematical concepts** to one another, applying them effectively in various problem situations.

This resource packet offers a valuable opportunity to deepen your understanding of mathematical patterns and to apply logical reasoning for justification and problem-solving using organized methods.

We are confident in your abilities and look forward to seeing you **excel** in this crucial phase of your journey toward excellence.

The Scientific Team for the National Science and Mathematics Olympiad (NSMO) – Mathematics Track

First Unit: ALGEBRA

Linear Equations in One Variable

To solve a one-variable linear equation, follow the steps below in order:

- **Eliminating Denominators (if any)**

If there are fractions in the equation, multiply each term by the least common multiple (LCM) of the denominators to eliminate them.

- **Expand parentheses**

Use the distributive property to simplify parentheses, such as

$$2(x+3) = 2x + 6$$

- **Move terms**

Move all the terms with the variable to one side and all the constants to the other using addition and subtraction.

- **Combine Like Terms**

Simplify both sides of the equation so that it takes the form:

$$ax = b \text{ ,} where\ a\ and\ b\ are\ constants.$$

- **Dividing by the variable's coefficient**

We divide both sides by a (the variable coefficient) to get the solution:

$$ax = b \Longrightarrow x = \frac{b}{a}$$

Note that:

- When $a \neq 0$ we have a unique solution: $x = \frac{b}{a}$
- When $a = 0$, $b \neq 0$ there is no solution.
- When $a = 0$, $b = 0$ any real number is a solution to the equation.

- **Check your answer**

We can check our answer by substituting it back into the original equation. If the original equation is not satisfied by our answer, then we made a mistake.

- **Note:** Sometimes we do not follow the exact order of the previous steps when a more direct method is available.

Example:

Solve the following equations:

$a)\ x + 3 = 4$ $\quad b)\ x - 2 = 9$ $\quad c)\ 7x = 49$

Solution:

$a)\ x = 1$ $\quad b)\ x = 11$ $\quad c)\ x = 7$

Exercises:

(1) Solve the following equations:

$a)\ \frac{x}{3} = 6$ $b)\ 2x - 1 = 19$

$c)\ 4x - 4 = x + 11$ $d)\ 9(x - 1) = 7(x + 1)$

(2) Solve:

$$\frac{1}{7}(5x + 2) = 1$$

(3) Solve:

$$\frac{1}{2}\left[\frac{1}{7}(5x - 1)\right] + 5 = 6$$

(4) **Word Problems:**

- If 7% of a number equals 56, what is the number?
- If the sum of four consecutive natural numbers is 50, what is the largest number
- Two natural numbers have a ratio of 2:3, and the difference between them is 14. What is the smaller number?
- Ahmed bought a computer at a discount of 35% off its original price, paying 1,300 riyals. How much was the original price of the computer?
- Mohammed bought a car and later sold it for 46,000 riyals, making a 15% profit. What was the purchase price of the car?

Challenge Problems:

(1) A train traveling at a constant speed was observed entering a 120-meter-long tunnel and exiting the tunnel at the same speed. The observation lasted one minute. If you know that the same train traveling at the same constant speed takes 20 seconds to pass through a traffic light in its entirety, how long is the train?

(2) We have a jar with a ratio of red balls to white balls of 1 to 4. When Saad replaces 2 white balls with 7 red balls, the ratio of red balls to white balls becomes 2 to 3. What is the ratio of the total number of balls now to the total number at the beginning?

(3) Leila gave birth to her first child on her 20th birthday, her second child exactly two years later, and her third child exactly two years after that. How old will Leila be when the combined ages of her three children equal her age?

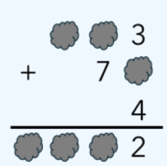
(4) A fruit basket contains apples and oranges. The ratio of apples to oranges in the basket is 3 to 8. We took one apple out of the basket. The ratio of apples to oranges in the basket became 1 to 3. How many oranges are in the basket?

(5) The arithmetic mean of six numbers is 4. When we added a seventh number, the new mean became 5. Find the seventh number.

(6) The coach was buying 17 balls for his sports club at a price of 481 riyals each. The salesman said, "You have to pay 10,177 riyals for the balls." How could he guess that he was being cheated without doing difficult calculations?

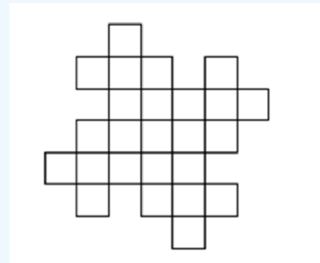
(7) Two numbers were multiplied together. Could the result be 20042401? (If so, give an example. If not, explain why).

(8) Suad wrote an addition exercise in which all the digits are different. After proudly completing it, her little brother spilled ink on her notebook. Can you help the girl rewrite her exercise?



(9) Color five squares in the figure below so that the remaining part can be divided into five congruent pieces.

(Providing one example is enough to answer the question).

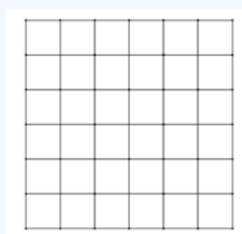


(10) A farmer bought three cows. Now each day he gets:

1) **From the first cow:** 2 liters of milk.
2) **From the second cow:** Half the amount of milk obtained from the third cow plus the same amount obtained from the first cow.
3) **From the third cow:** Half of the total amount of milk obtained from all three cows.

How many liters of milk does the farmer get in total each day?

(11) Color six squares in the grid below black so that it becomes impossible to find any white strip of size 1 × 6 or any white square of size 3 × 3 after coloring



(12) In the figure below, the addition of three numbers is shown.

Each letter represents one of the digits from 0 to 9 and stands for the same digit each time it appears.

Different letters represent different digits.

The leftmost digit of the result is not 0.

Find all possible values of the sum.

$$\begin{array}{r} JMO \\ JMO \\ +\ JMO \\ \hline IMO \end{array}$$

Multi-Variable Linear Equations

The general form of a system of two linear equations in two variables is:

$$\begin{cases} a_1x + b_1y = c_1 & (1) \\ a_2x + b_2y = c_2 & (2) \end{cases}$$

Where $a_1, b_1, c_1, a_2, b_2, c_2$ are real numbers

We have two linear equations in two variables, and they represent two straight lines in the coordinate plane.

The solution of the system corresponds to the point(s) of intersection of these two lines (since such a point satisfies both equations).

To eliminate one of the variables and solve the system, we use:

(i) the usual algebraic operations.

(ii) the substitution method.

In many cases, method (i) is more efficient.

When

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

- Thus, the two lines intersect at exactly one point, and therefore the system has a unique solution, which is

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \quad \square \quad y = \frac{a_1c_2 - a_2b_1}{a_1b_2 - a_2b_1}$$

When

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

- Then the two lines coincide, and therefore all their points are common. As a result, the system has infinitely many solutions

When

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

- Then the two lines are parallel and have no points of intersection; therefore, the system has no solution.

Example:

How many solutions does each of the following systems have?

$$(a)\ \begin{cases} 5x + 2y = 4 \\ 7x + 5y = 1 \end{cases}$$

$$(b)\ \begin{cases} 4x + 3y = 3 \\ 8x + 6y = 7 \end{cases}$$

Solution:

(a) One solution:

$$\frac{a_1}{a_2} = \frac{5}{7} \neq \frac{b_1}{b_2} = \frac{2}{5}$$

(b) There is no solution:

$$\frac{a_1}{a_2} = \frac{4}{8} = \frac{b_1}{b_2} = \frac{3}{6} \neq \frac{c_1}{c_2} = \frac{3}{7}$$

Exercises:

(1) How many solutions are there for this system?

$$\begin{cases} x + 5y = 3 \\ 2x + 10y = 6 \end{cases}$$

(2) Solve the following system of equations:

(a) $\begin{cases} x + y = 13 \\ x - y = 7 \end{cases}$ (b) $\begin{cases} 2x + y = 9 \\ x + 4y = 8 \end{cases}$

(c) $\begin{cases} x + y = 14 \\ x - y = 2 \end{cases}$ (d) $\begin{cases} 2x - y = 6 \\ x + 4y = 21 \end{cases}$

(3) Two numbers have a sum of 42 and a difference of 8. What are the two numbers?

(4) At a certain time, Reema notices that her digital watch reads a minutes after two o'clock. Fifteen minutes later, it reads b minutes after three o'clock. She is amused to note that a is six times b. What time was it when she looked at her watch for the second time?

Challenge Problems:

(1) Solve the following system of equations:

$$\begin{cases} \frac{1}{a} - \frac{1}{b} = 6 \\ \frac{1}{a} + \frac{1}{b} = 14 \end{cases}$$

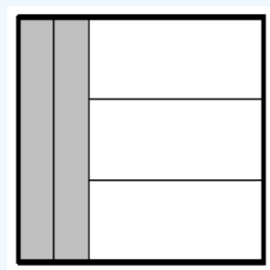
(2) Solve the following system of equations:

$$\begin{cases} x + y = 8 \\ y + z = 4 \\ z + x = 6 \end{cases}$$

(3) Give an example of two fractions whose difference is equal to their product.

(4) The figure shown is a square composed of five rectangles, all having the same perimeter.

What is the ratio of the area of one shaded rectangle to the area of one unshaded rectangle?



(5) We want to divide a certain amount of money equally among a group of children. If each child receives 60 halalas, 2.10 riyals will remain. However, if each child receives 20 halalas more than this amount, there will be just enough for each child to receive 70 halalas. How many children are in the group?

(6) The number 3600 can be written as $2^a \times 3^b \times 4^c \times 5^d$. where $a, b, c, and\ d$ are positive integers.

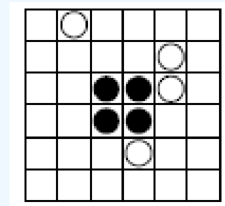
If $a + b + c + d = 7$. what is the value of c ?

(7) If you can remove ten digits from the number 12345123451234512345. What is the largest number you can get?

(8) Cut the following shape into four identical parts

(both in shape and area)

so that each part contains one black circle and one white circle.



(9) My brother has four children. They are $5, 8, 13, and\ 15$ years old. Their names (in no particular order) are Muhammad, Raja, Najah, and Nour. One of the girls is in kindergarten, Raja is older than Muhammad, and the sum of Raja and Nour's ages is divisible by three.

Is Nour a boy or a girl?

(10) There are one hundred people living on an island. Some of them always lie, while others always tell the truth, and each of them has a favorite season of the year. Every person was asked the following four questions:

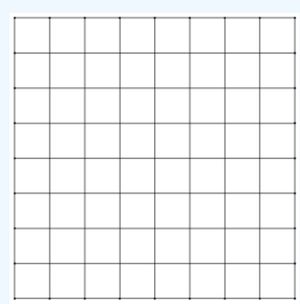
- **Do you like winter?**
- **Do you like spring?**
- **Do you like summer?**
- **Do you like fall?**

There were 25 "yes" answers to the first question, 25 "yes" answers to the second, 45 "yes" answers to the third, and 55 "yes" answers to the fourth.

How many liars are there on the island?

(11) We have the following 8×8 table divided into 64 small squares.

Can you color 17 small squares black so that no two squares share an edge or even a corner?



Percentages

Percent literally means "per hundred." Percent is just a shorthand way of writing the ratio of a number to 100.

For example, 54% means 54 out of 100 or

$$\frac{54}{100}$$

Specifically, if a is x percent of b, then

$$\frac{a}{b} = \frac{x}{100}$$

For example, 3 out of 16 is written as a percentage as follows

$$\frac{3}{16} = \frac{x}{100}$$

By simplifying, we obtain

$$x = 18\frac{3}{4}$$

Final

Answer: 3 is 18,75% of 16

- If a number, x , is increased by $k\%$, then the result is:

$$x \cdot \left(1 + \frac{k}{100}\right)$$

- Similarly, if x is decreased by $k\%$, then the result is:

$$x \cdot \left(1 - \frac{k}{100}\right)$$

Examples:

When a number is reduced by 40%, the result is 36. What is the original number?

Solution:

Let our number be x . We have

$$x\left(1-\frac{40}{100}\right)=36$$

$$\Rightarrow \frac{6}{10}x=36$$

$$\Rightarrow x=60$$

Exercises:

(1) For all nonzero x, y, and z, find the value of k such that:

$$\frac{7}{x+y}=\frac{k}{x+z}=\frac{11}{z-y}$$

(2) Find the values of $a, b, c \in N$ that satisfy the following:

$$\frac{a+1}{3}=\frac{b+2}{2}=\frac{8}{c+3}$$

(3) 36 is 120% of what number?

(4) What is the discount percentage for an item if its price changes from 2890 riyals to 2023 riyals?

(5) If Nora brought 60 cookies to school, gave 40% of them to her teachers, 25% of the rest to her friends, and ate one-third of what was left, how many cookies are left?

(6) If the ratio of the interior angles of a pentagon is $2:3:4:5:6$, what is the measure of the largest angle?

(7) If b is 5% greater than a, and b is 15% less than c,
What is the ratio of a to c ?

(8) A rectangle has been increased in length by 50% and in width by 20%. What is the percentage increase in its area?

(9) Every day, 20% of the fish are sold at the fish market. If 2000 fish remained at the end of Tuesday, how many fish were there at the beginning of Monday?

(10) If t is 25% of u , then what percent of $4t$ is $2u$?

(11) Given that

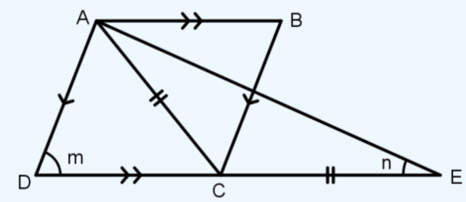
$$yz:zx:xy=1:2:3 \qquad and \qquad \frac{x}{yz}:\frac{y}{zx}=1:k$$

Find k .

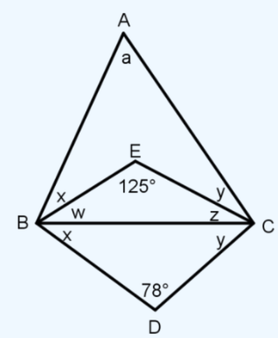
Second Unit: Geometry

Revision Exercises

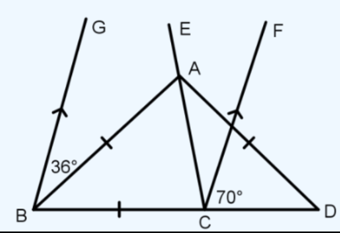
(1) In the following diagram $ABCD$ is a rhombus, ACE is an isosceles triangle with $AC = CE, D, E, C$ are on the same line. If $\angle AEC = n, \angle ADC = m$. Write an equation the relates m, n.



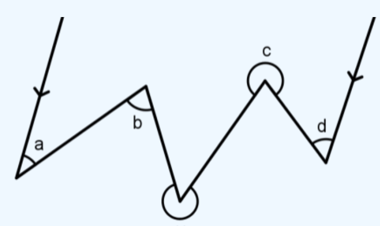
(2) In the following diagram $ABCD$ is a quadrilateral. Point E lies inside such that $\angle ABE = \angle DBC = x, \angle ACE = \angle DCB = y$. If $\angle BEC = 125°, \angle BDC = 78°$ Find $\angle A$.



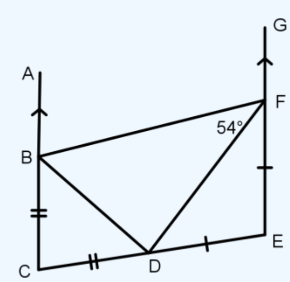
(3) In the following diagram: given $\underline{AB} = \underline{BC} = \underline{AD}$ Find $\angle EAD$.



(4) Find the value of x in terms of a, b, c, d.



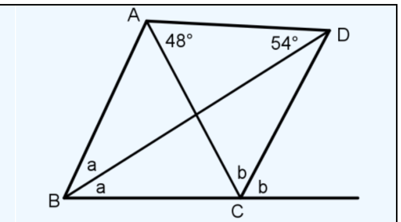
(5) In the following diagram we have $\underline{BC} = \underline{CD}, \underline{DE} = \underline{EF}, AC \parallel GE$. Given $\angle BFD = 54°$ Find $\angle DBF$.



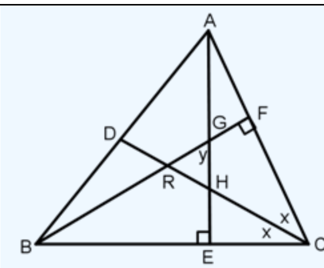
(6) Given the information in the diagram:

(a) Find $\angle BAC, \angle BDC$ in terms of a, b.

(b) if $a = \frac{2}{3}b$. Find a, b.

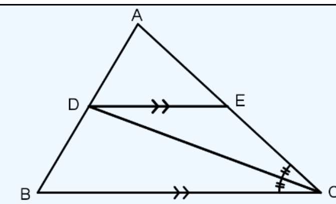


(7) In the following diagram: give an equation that related x, y.

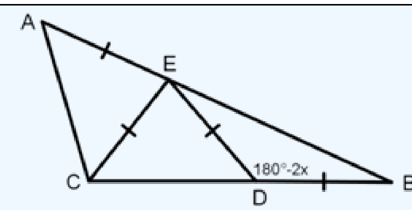


(8) in the following diagram: CD bisects $\angle ACB, \angle ACB = 40°, \angle B = 70°$,

$DE \parallel BC$. Find the measure of $\angle EDC, \angle BDC$.

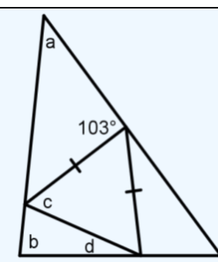


(9) In triangle ΔABC, E, D lie on $\underline{AB}, \underline{BC}$ such that $\underline{AE} = \underline{EC} = \underline{DE} = \underline{DB}$. If $\angle EDB = 180° - 2x$. Find $\angle ACE$ in terms of x.



(10) In the following figure:

If $a = \frac{2}{3}c = \frac{1}{2}b = 3d$. Then find a, b, c, d.



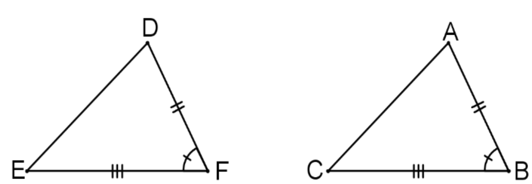
Triangle Congruence

If we draw two triangles, let's say the side lengths of the first are 4,5,7, and then we draw the second triangle using the exact same side lengths on a transparent sheet.
Placing the transparent sheet (which holds the second triangle) directly over the sheet containing the first triangle, we would observe that the two triangles perfectly coincide (superpose). We would also notice that all of the corresponding angles are equal in measure.
This phenomenon is what we call triangle congruence. The example mentioned here illustrates one specific case of congruence, and we will now proceed to discuss the cases of congruence in detail

Cases of Triangle Congruence

First Case:

Two triangles are congruent if two sides and the included angle of the first triangle are congruent (equal in measure) to the corresponding two sides and the included angle of the second triangle. We will refer to this case of congruence using the abbreviation SAS. (side-angle-side)



In triangles $\triangle ABC, \triangle DFE$, if:

$$\{AB = DF\ BC = FE\ \angle B \cong \angle F$$

Then $\triangle ABC \cong \triangle DFE\ (SAS)$

And thus:

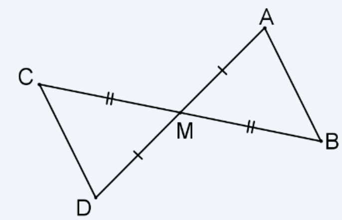
$$\{AC = DE\ \angle A \cong \angle D\ \angle C \cong \angle E$$

Exercise (1):

In the following figure, M is the midpoint of AD and BC. Show that:

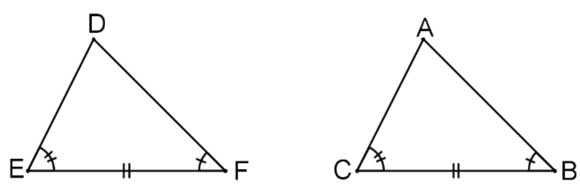
a) $AB = CD$

b) $AB \parallel CD$



Case two:

Two triangles are congruent if two angles and the included side (the side connecting the vertices of the two angles) in one triangle are congruent (equal in measure and length) to the corresponding two angles and included side in the other triangle. We will call this SAS.



In triangles ABC, DFE, if:

$$\{CB = EF \ \angle C \cong \angle E \ \angle B \cong \angle F$$

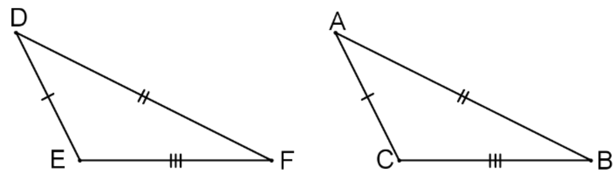
Then $\triangle ABC \cong \triangle DFE (ASA)$

Moreover, we get that:

$$\{AB = DF \ AC = DE \ \angle A \cong \angle D$$

Case Three:

Two triangles are congruent if all three sides of the first triangle are congruent (equal in length) to the corresponding three sides of the second triangle. We will refer to this by SSS.



In triangles ABC, DFE, if:

$$\{AB = DF\ AC = DE\ BC = EF$$

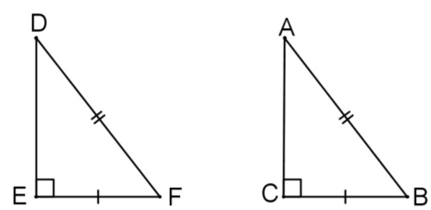
Then $\triangle ABC \cong \triangle DFE (SSS)$.

Moreover, we get that:

$$\{\angle A \cong \angle D\ \angle C \cong \angle E\ \angle B \cong \angle F$$

Forth Case:

Two right-angled triangles are congruent if the hypotenuse and one leg (side forming the right angle) of the first triangle are congruent (equal in length) to the corresponding hypotenuse and one leg of the second triangle. We will refer to this by HS (hypotenuse-side).



In Triangles ABC, DFE, If:

$$\{AB = DF\ CB = EF\ m(\angle C) = m(\angle E) = 90°$$

Then $\triangle ABC \cong \triangle DFE (HS)$,

Moreover, we get that:

$$\{AC = DE\ \angle A \cong \angle D\ \angle B \cong \angle F$$

Remarks:

The two congruent sides in an isosceles triangle are called the legs. The third side, which is not congruent to the others, is called the base.

Isosceles Triangle Theorem:

The angles of the base of an isosceles triangle are equal. And the opposite is true, if two angles of a triangle are equal, then the triangle is isosceles.

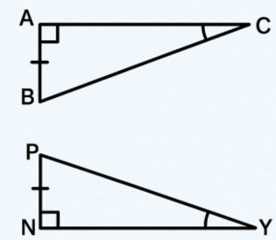
Conclusions:

- The angles of an equilateral triangle are all equal,
- And there measure is 60^o degrees.
- The angle bisector of the vertex of an isosceles triangle is perpendicular to the base and also bisects the base.
- The triangle with equal angles is also an equilateral triangle (has equal sides).

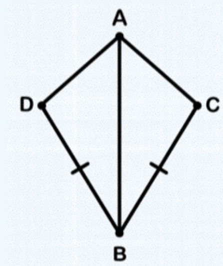
Exercises:

In the following figures, find (if it exists) the triangle that is congruent to ABC and state the postulate you used.

(1)



(2)



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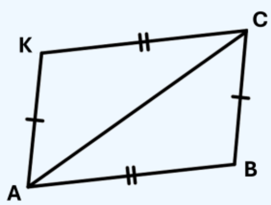

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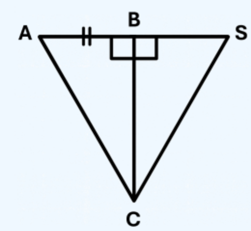

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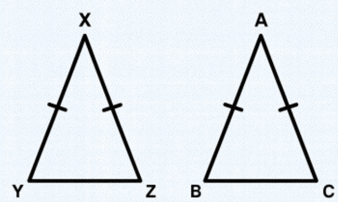

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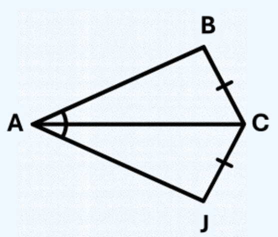

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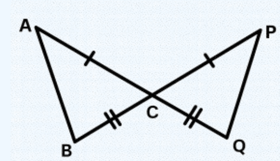

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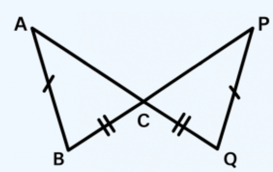

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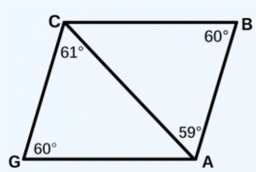

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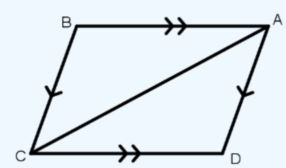

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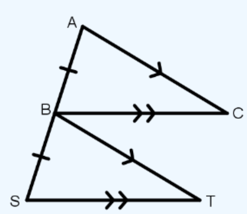

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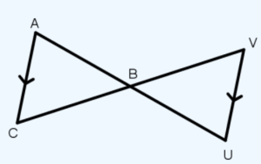

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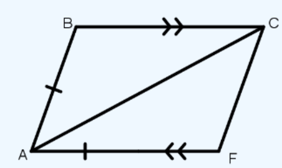

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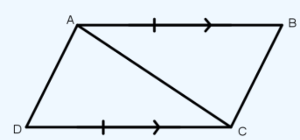

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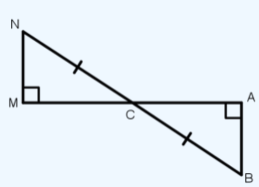
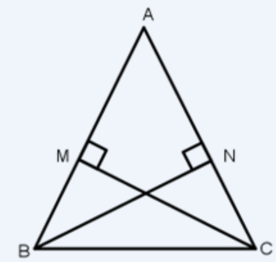
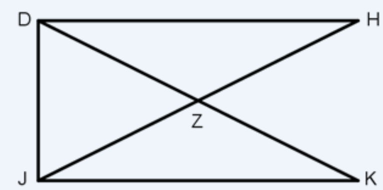


(16) In the adjacent figure: if $\overline{AB} \cong \overline{AC}$, and $\overline{BN} \perp \overline{AC}$, $\overline{CM} \perp \overline{AB}$, explain how you can prove that $\Delta ABN \cong \Delta ACM$

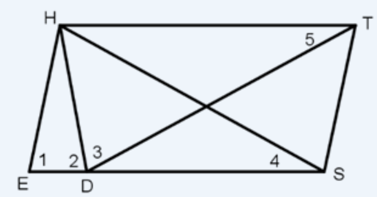


(17) In the adjacent figure: if $JH = DK$, $DH \perp DJ$, $JK \perp DJ$ prove that $\angle H = \angle K$

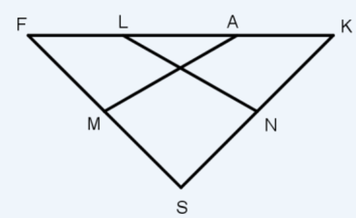


(18) In the adjacent figure: if $ES = DT$, $\angle 1 = \angle 2 = \angle 3$,

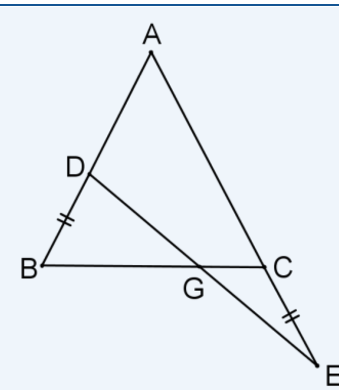
prove that $\angle 4 \cong \angle 5$



(19) In the adjacent figure: if $FL = AK$, $SF = SK$, and M is the midpoint of SF, N is the midpoint of SK, prove that $AM = LN$.



(20) In the triangle $\triangle ABC$, $AB = AC$, point D lies on AB, and point E lies on the extension of AC such that $BD = CE$. If DE intersects BC at G, prove that $DG = GE$.



Third Unit: Number Theory

Even and Odd Integers

Integers can be divided into two types of numbers: those that are divisible by 2 are called **even** numbers, and the rest of the numbers that are not divisible by 2 are called **odd** numbers. An even integer can be written in the form $2k$, where k is an integer. An odd integer can also be written in the form $2k + 1$, where k is an integer. Any given integer is exclusively either even or odd; it cannot be both.

Other Properties of Even and Odd numbers:

(1) Odd ≠ Even.

(2) Odd + Even = even +odd = odd, odd-even = even – odd = odd.

(3) even ± even= odd± odd= even.

(4) If the multiplication of two integers is even, then one of them has to be even.

(5) The multiplication of two consecutive integers has to be even.

(6) If the sum (or difference) of some integers is odd, then the number of odds in those numbers is odd.

(7) If the sum (or difference) of some integers is even, then the number of odds in those numbers is even.

(8) if the multiplication of some integers is odd, then all of those integers must be odd.

(9) if the multiplication of some integers is even, then at least one of those integers must be even.

Example 1:

Divide the following numbers into even and odd: $102, 203, 519, 3340, 70015, 87654$. Then find a pattern for even and odd numbers.

Solution:

- The number 102 is even since we can write it as $102 = 2(51)$.
- 203 is odd since $203 = 2(101) + 1$.
- 519 is odd since $519 = 2(259) + 1$.
- 3340 is even since $3340 = 2(1670)$.
- $70015 = 2(35007) + 1$ so it is an ___ number.
- $87654 = 2(43827)$ so it is an ___ number.

If the units digit is $0,2,4,6,8$ then the number is ___.

If the units digit is $1,3,5,7,9$ then the number is ___.

Example 2:

If we add two even numbers, then the result would be:

(a) Even (b) Odd

Solution:

Answer is (a) Since we know from the properties that the sum of two even numbers is even. Moreover, we can check that the units digit if we add two of $0,2,4,6,8$ then the resulting number would still have a $0,2,4,6,8$ as a units digit.

Can you use the same logic to determine the following:

If we add two odd numbers, then the result would be:

(a) Even (b) Odd

If we add an even and an odd number, then the result would be:

(a) Even (b) Odd

Example 3:

If we pick three random numbers a, b, c then the numbers:

$$\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2}$$

(a) all are non-integers

(b) at least one of them is an integer.

(c) At least two of them are integers

(d) All of them are integers.

Solution:

The answer is (b). We can see that since we have two cases:

Case 1: All of the numbers have the same parity (all odd or all even), which means that sum of any two is even so it is divisible by 2. This leads to $\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2}$ being all integers.

Case 2: One of them has a different parity (one odd and two even or one even and two odd). And if we do this case (For example a, c are odd while b is even). We can see that two sums ($a+b, b+c$ are odd, while $a+c$ is even). Thus, only one of $\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2}$ is an integer. Following the same logic. The other case should also result in one number being an integer, which gives the answer (b).

Example 4:

If you know that the sum of 100 positive integers is 10000, and that the number of odd numbers (in the 100 numbers) is more than the number of even numbers. Find the largest number of even numbers in the set.

Solution:

Assuming that the number of even numbers is x and that the number of odd numbers is y. Then we have:

$$x + y = 100, y > x$$

But since we know that the sum is even, so y has to be even (look at the properties of even and odds). Thus, y is at least 52, and that means that x is at most 48. It suffices to give an example which is: $\underbrace{1 + 1 + \ldots + 1}_{52 time} + \underbrace{2 + 2 + \ldots + 2}_{47 time} + 9854 = 10000$

Thus, the answer is 48.

Example 5:

We have n numbers: $x_1, x_2, x_3, \ldots, x_n$ all of them are either $1, -1$. If:

$$x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n + x_nx_1 = 0$$

Then what can we say about n?

(a) even (b) odd (c) n is multiple of 4 (d) cannot say anything about n.

Solution:

The answer is (c). Notice that the sum:

$$x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n + x_nx_1$$

Contains n terms each of which is either 1 or -1. Thus, n has to be even so that the sum is 0. Now we can write $n = 2k$, this means that the numbers $x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1$ have k of them equal to 1 while the other k are equal to -1. But notice that their multiplication is:

$$(x_1x_2).(x_2x_3), \ldots .(x_nx_1) = x_1^2x_2^2 \ldots x_n^2 = 1$$

Thus:

$$(-1)^k(1)^k = 1$$

This means that k should also be even. Therefore, $k = 2m \Rightarrow n = 2k = 4m$ which gives us that n is a multiple of 4.

Exercises:

1) Find the parity of the number $\frac{12214509875435678863333454214938}{2}$ without finding the value of the number.

2) We have 50 books that we would like to put in 5 boxes. Can we divide the books into the boxes so that each box has an odd number of books? Explain your answer

3) If you knew that a, b are two consecutive integers, $c = ab$, $N^2 = a^2 + b^2 + c^2$. Determine the parity of the number N (odd or even) and show your reasoning.

4) Challenge: in the following pattern $1,2,5,13,34,89, \dots$ in any three consecutive numbers, the sum of the first and third numbers it is equal to three times the number in the middle. For example, $1,2,5$ we have $1 + 5 = 3(2)$. Or $2,5,13$ we have $2 + 13 = 3(5)$ and so on. What is the parity of the 2003^{th} number?

5) Given three integer numbers x, y, z two of which are odd and the third is even. Show that:

$$(x + 1)(y + 2)(z + 3)$$

Has to be even.

6) If you can put wither $' + '$ or $' - '$ signs between any two numbers in the list:

$$1 \quad 2 \quad 3 \quad 4 \quad \dots\dots \quad 2017$$

Will the result be positive or negative?

7) (a) We have the line

$$1 * 2 * 3 * 4 * 5 * 6 * 7 * 8 = 0$$

Can we replace the stars with $+, -$ to make the statement correct?

(b) Same questions, but for the statement:

$$1 * 2 * 3 * 4 * 5 * 6 * 7 * 8 * 9 = 0$$

8) In a chess competition, each player will play 20 games. Wins add 3 points to the total score, draws add one point to the score, and defeats subtract one point from the total score. Can a player get a final score of 39? Show your reasoning.

9) (a) There are 10 baskets on a circle. Can we put oranges in each basket so that the difference in the number of oranges between each consecutive baskets is 1?

(b) What if there was 9 baskets instead? Show your reasoning.

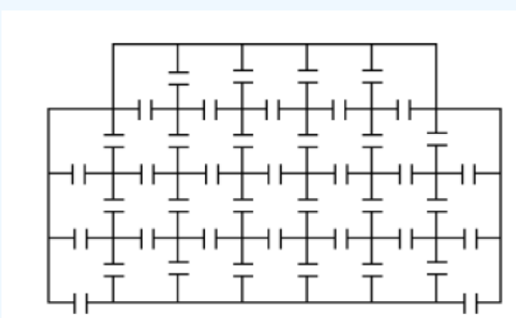
10) A worm move on a straight line, she can jump 6 or 8 cm in any of the directions (left or right). Can the worm reach to a point that is:

(a) $1.5\ cm$ (b) 7 cm (c) $4\ cm$

Far away from its original position?

11) Challenge: how can we arrange the 10 numbers 1,1,2,2,3,3,4,4,5,5 in a line so that there is one number between 1,1, two numbers between 2,2, three numbers between 3,3, four numbers between 4,4, and 5 numbers between 5,5?

12) Challenge: How can we visit the 26 rooms in the diagram such that we enter each room exactly one time?



Forth Unit: Combinatorics

Continuation of Counting Principles

First: The Two Fundamental Counting Principles

Let's start with this straightforward example.

Example:

Mohammed and Ali went to a sporting goods store that sells four different types of shoes and seven different types of athletic gloves. Mohammed has enough money to buy one shoe AND one glove. Ali has enough money to buy either one shoe or one glove. The question is:

(A) In how many different ways can Mohammed buy a shoe and a glove?

(B) In how many different ways can Ali buy a shoe or a glove, and not both?

Solution:

A) $4 \times 7 = 28$

B) $4 + 7 = 11$

Many counting problems use one of these two principles, which is the reason behind the name: **Fundamental Counting Principles**. Let's learn about them and see the creative ways to apply them in counting problems.

The Multiplication Principle:

If the Event A can occur in m different ways and Event B can occur in n different ways, and the two events are independent, then the two events can occur together in $m \cdot n$ different ways.

Examples:

(1) In how many ways can we arrange 5 different books on a shelf?

Solution:

1st Position	2nd Position	3rd Position	4th Position	5th Position
5 ways	4 ways	3 ways	2 ways	One way

When choosing a book to place in the 1st position of the shelf, we will have 5 ways because we own 5 different books. After we place a book in the 1st position, we will have 4 ways to choose the book we place in the 2nd position, because the number of books has decreased after placing one of them in the 1st position. In the same way, we will calculate the number of ways for each of the 5 positions. Then, we use the Multiplication Principle to find the total number of arrangements. Thus, the number of ways is $= 5 \times 4 \times 3 \times 2 \times 1$, and this equals 120 ways.

(2) In how many ways can we select 4 cards that are different in color and different in value (the number written on them), given that we have 52 cards divided into 4 colors, and the cards of each color are numbered with the values from 1 to 13?

Solution:

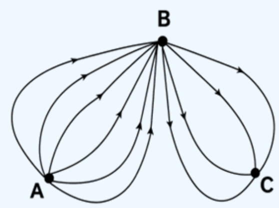
We are required to choose one card of each color, each with a different number

First Color	Second Color	Third Color	Fourth Color
13	12	11	10

We have 13 ways to select a card of the First Color. When selecting a card of the Second Color, we must exclude the card that carries the same number as the first card, and thus, the number of ways will decrease to 12 ways. In the same way, we will have 11 ways to select the card of the Third Color and 10 ways to select the card of the Fourth Color. Then, we use the Multiplication Principle to find the total number of ways. Thus, the number of ways is $=$ $13 \times 12 \times 11 \times 10$

Exercises:

(1) There are three villages, A, B, and C, in a country. One can travel from village A to village B in 6 ways, and from village B to village C in 4 ways for the journey, as shown in the figure. In how many different ways can one travel from A to C?



(2) How many natural numbers consisting of three digits are there?

(3) In how many ways can 5 students be arranged in a row, provided they are selected from a group of 8 students?

(4) In how many ways can one enter a complex through one door and exit through a different door, given that the complex has ten doors?

(5) We want to arrange 4 girls and 5 mothers in a row, provided that the girls must remain next to each other. In how many ways can this be done?

The Addition Principle: If event A can occur in m different ways and event B can occur in n different ways, and the two events are mutually exclusive, then the number of ways in which A **or** B can occur is $m + n$.

We use this principle when solving a problem that requires dividing it into several mutually exclusive cases, as in the previous example.

Example:

(1) If we have 5 different books on a shelf, in how many ways can we arrange some (or all) of the books in a stack? The stack may contain one book only or more.

Solution:

We have several cases:

- **Case 1:** The stack contains 1 book → number of ways = 5
- **Case 2:** The stack contains 2 books → number of ways $= 5 \times 4 = 20$
- **Case 3:** The stack contains 3 books → number of ways $= 5 \times 4 \times 3 = 60$
- **Case 4:** The stack contains 4 books → number of ways $= 5 \times 4 \times 3 \times 2 = 120$
- **Case 5:** The stack contains 5 books → number of ways $= 5 \times 4 \times 3 \times 2 \times 1 = 120$

Using the Addition Principle, the total number is $= 5 + 20 + 60 + 120 + 120 = 325$ ways.

Exercises:

(6) In how many ways can we choose a pen or a notebook from a collection that contains 10 different notebooks and 7 different pens?

(7) Salma owns 5 different colors of tea cups, 3 different colors of serving plates, and 4 different colors of tea spoons. In how many different ways can she choose two items of two different types of utensils?

(8) In a first-grade class, there are 12 girls, and 3 of them have names that start with the letter "A." In how many ways can 4 girls be arranged in a line if only one of them has a name starting with "A"?

(9) Al-Mawahib School has three classrooms: the first has 20 students, the second has 13 students, and the third has 8 students. In how many different ways can two students from two different classes be chosen to work together on a task?

(10) You can travel from city A to city B using one of two land routes or one of three air routes. You can then travel from city B to city C using one of four land routes or one of five air routes.

(a) How many travel routes are there from city A to city C through city B?

(b) How many routes are there if both routes used are by air?

(c) How many routes are there if both routes used are by land?

(d) How many routes are there if one route is by land and the other by air?

Second: Counting Numbers and Strings

- **A natural number with n digits:** We say that a natural number consists of one or more digits (places) if its leftmost nonzero digit is not zero.

For example: The number 2345 consists of four digits, but the number 034 is not considered a three-digit number, because the leading zero does not count. It is written as 34, meaning the number has two digits only.

- **A string of digits:**

We say that a string is made up of one or more digits, regardless of whether the leftmost digit is zero or not.

For example: each of the following: 01234, 00034, and 12302 is a string made up of five digits.

- **Palindromic numbers:**

A natural number is called palindromic (symmetric) if reading its digits from right to left gives the same sequence as reading them from left to right.

For example: 1234321, 919, and 6 are palindromic numbers, while 2342 is not.

The counting principles will help us determine the number of such numbers or strings in a simple yet profound way. Let's begin this creative journey!

Exercises:

(11) How many positive integers with four different digits are there?

(12) How many positive odd integers consist of five digits?

(13) How many two-digit numbers can be formed from the digits of the set $\{0,3,4,5,6,7,8,9\}$, provided that the sum of their digits is odd?

(14) How many numbers between 0 and 1000 contain the digit 5 exactly once?

(15) How many three-digit numbers that are divisible by 3 and have distinct digits can be formed from the digits $\{1, 3, 7, 8, 9\}$?

(16) How many 10-digit strings consisting of only 0s and 1s contain exactly five consecutive zeros?

(17) How many five-digit numbers satisfy the condition that the sum of the first and fifth digits equals 5?

(18) How many six-digit numbers are there in which all digits are of the same type (either all even or all odd)?

(19) How many positive four-digit numbers contain exactly one digit "1" and exactly one digit "3"?

(20) How many three-digit numbers have at least one even digit?

(21) How many four-digit numbers are there such that no two even digits are adjacent?

(22) How many positive integers less than 1000 do not contain the digit 7?

(23) How many positive five-digit numbers with distinct digits satisfy the condition that the positive difference between the first and last digit equals 2?

(24) When writing all the numbers from 1 to 100, how many times do we write the digit 6?

(25) How many palindromic numbers are there that have:

(a) 6 digits, formed from the digits $\{1,2,3,4,5,6,7,8\}$.

(b) 7 digits, formed from the same set of digits above.

(c) 5 digits, formed from the digits $\{0,1,2,3,4,5,6,7,8\}$.

Third: Counting Words

We can calculate the number of English "words" made up of n letters using the Multiplication Principle directly (the "words" do not have to have any actual meaning).

Example: The number of words consisting of 4 letters is:

1st letter		2nd letter		3rd letter		4th letter	
Any letter		Any letter		Any letter		Any letter	**Number of Words**
26 ways	×	26 ways	×	26 ways	×	26 ways	$= 26^4$

Exercises:

(26) How many words can be formed from 5 different letters?

(27) In how many ways can the letters of the word PRODUCT be arranged?

(28) How many 6-letter words can be formed if the 1st, 3rd, and 6th letters are different?

(29) How many 3-letter words contain the letter M exactly once?

(30) How many 3-letter words contain the letter M at least once, if no other letter is repeated?

(31) How many 3-letter words contain the letter M at least once, if repetition of other letters is allowed (that is, another letter may appear twice)?

(32) The Saudi team for the Cybersecurity Olympiad invented their own language consisting of only four letters. In their language, the longest word contains at most 6 letters. How many words are in this language?

(33) We define a good word as a word made up only of the letters A, B, and C (it's allowed for any one of these letters not to appear) such that B does not immediately follow A, M does not immediately follow B, and A does not immediately follow C. How many 7-letter good words are there?

(34) How many palindromic words of length seven are there that do not begin with a vowel? The vowels are (A, E, I, U, Y).

(35) In how many ways can we write all the English letters in a row so that there are exactly five letters between the two letters x and y?

Solutions

Algebra Solutions

One-Variable Linear Equations:

Exercises:

(1)

$$(a)x = 18, (b)x = 10, (c)x = 5, (d)x = 8$$

(2)

$$\frac{1}{7}(5x + 2) = 1$$

$$\Longrightarrow 5x + 2 = 7$$
$$\Longrightarrow 5x = 5$$
$$\Longrightarrow x = 1$$

(3)

$$\frac{1}{2}\left[\frac{1}{7}(5x - 1)\right] + 5 = 6$$

$$\Longrightarrow \frac{1}{2}\left[\frac{1}{7}(5x - 1)\right] = 1$$

$$\Longrightarrow \frac{1}{7}(5x - 1) = 2$$

$$\Longrightarrow 5x - 1 = 14$$

$$\Longrightarrow 5x = 15$$

$$\Longrightarrow x = 3$$

(4)

(a) Let our number be x . We have

$$\frac{7}{100}x = 56 \Rightarrow x = 56 \times \frac{100}{7} = 800$$

(b) Let the numbers be $n, n+1, n+2, n+3$.

$$n + (n+1) + (n+2) + (n+3) = 50 \Rightarrow 4n + 6 = 50 \Rightarrow n = 11$$

So, the largest number $n + 3 = 14$

(c) Let the numbers be $2k$و$3k$.

$$3k - 2k = k = 14 \Rightarrow k = 14 \Rightarrow (2k, 3k) = (28,42)$$

(d) let P the original price. We have

$$\frac{65}{100}P = 1300 \Rightarrow P = 1300 \times \frac{100}{65} = 2000$$

(e) Let c be the purchase price

$$\frac{115}{100}C = 46000 \Rightarrow C = 46000 \times \frac{100}{115} = 40000$$

Challenge Problems:

(1)

Let the length of the train be x meters.

Since it takes **60 seconds** to completely pass through a tunnel that is **120 meters** long, starting from the moment the train enters until it fully exits, the constant speed of the train is:

$$\frac{120+x}{60}$$

The same train takes **20 seconds** to completely pass a signal post at the same speed, which equals:

$$\frac{x}{20}$$

Hence,

$$\frac{120+x}{60}=\frac{x}{20}$$

Solving this equation gives $x=60$,

which means that the length of the train is **60 meters**.

(2)

Let the number of red balls at the beginning be n, then the number of white balls at the beginning is $4n$.

After replacing **2** white balls with **7** red ones, the number of red balls becomes $(n+7)$, and the number of white balls becomes $(4n-2)$.

so, we can write the equation:

$$\frac{n+7}{4n-2}=\frac{2}{3}$$

Simplifying gives:

$$3(n+7)=2(4n-2)3n+21=8n-4n=5$$

Therefore, the total number of balls at the beginning is:

$$n+4n=5n=25$$

The total number of balls at the end is:

$$(n+7)+(4n-2)=5n+5=30$$

Hence, the required ratio of the total numbers is: $30:25=6:5$

(3)

Let the base year be the year when **Laila** was **20 years old**.

Suppose that after x **years**, Laila's age will be equal to the **sum of the ages of her three children**.

At that time, Laila's age will be $x + 20$ years,

and the ages of her three children will be x, $x - 2$, and $x - 4$ years respectively.

Hence, we can form the equation:

$$x + 20 = x + (x - 2) + (x - 4)$$

Simplifying:

$$x + 20 = 3x - 6 \Rightarrow 26 = 2x \Rightarrow x = 13$$

Therefore, Laila's age at that time will be: $20 + 13 = 33\ years$.

(4)

Let there be $3x$ **apples** and $8x$ **oranges** in the basket at the beginning.

After removing one apple, the number of apples in the basket becomes $3x - 1$,

while the number of oranges remains $8x$.

The ratio of apples to oranges is now **1 : 3**,

so we can form the equation:

$$\frac{3x - 1}{8x} = \frac{1}{3}$$

Solving for x gives:

$$x = 3$$

Therefore, the number of oranges in the basket is:

$$8x = 8(3) = 24$$

Hence, there are **24 oranges** in the basket.

(5)

The sum of six numbers is **24**,

since $4 \times 6 = 24$.

The sum of seven numbers is **35**,

since $5 \times 7 = 35$.

Therefore, the seventh number is: $35 - 24 = 11$

(6)

Since **17** is less than **20**, and **481** is less than **500**,

the product $20 \times 500 = 10{,}000$.

Therefore, the actual cost of the balls is **less than 10,000**.

(7)

It is not possible.

If one of the two numbers is even, then the product of the two numbers will be even.

When this product is multiplied by the sum of the numbers (whether that sum is even or odd),

the final product will still be even.

However, if both numbers are odd, their sum will be even, and their product will be odd.

When an odd number is multiplied by an even number, the result is even.

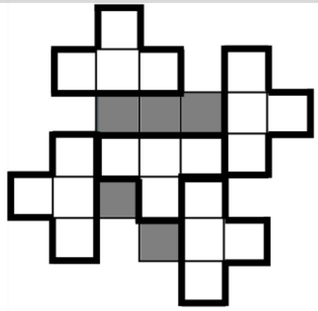
Therefore, in every possible case, the final result will be **even**,

whereas **20042401** is **odd**.

(8)

$$\begin{array}{r} 983 \\ +\quad 75 \\ 4 \\ \hline 1062 \end{array}$$

(9)



(10)

The third cow produces half the amount of milk.

Therefore, the second cow produces one quarter of the milk,

which equals 2 liters more than what the first cow produces.

So, $2 + 2 = 4$ liters form one quarter of the milk.

This means the second cow produces $4 + 2 = 6$ liters,

and the third cow produces 8 liters of milk.

In conclusion:

the first cow produces **2 liters**,

the second cow **6 liters**,

and the third cow **8 liters**,

giving a total of **16 liters** of milk.

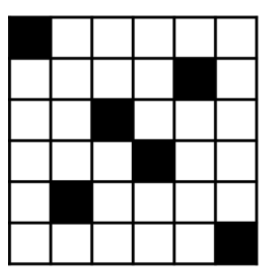
(11)

To ensure that there is no white strip of size 1×6, no row or column should be completely white.

By placing black squares as shown in the grid,

we also make sure that there is no white square block of size 3×3 after coloring.

One possible solution is shown below:



(12)

Let's first determine the possible values of **O**.

Since the number **3O** has a units digit of **O**, the possible values for **O** are **0** or **5**.

– If **O = 0**, there is no carryover to the tens' column.

Therefore, we need to determine the value of **M**.

The number **3M** must also have a unit's digit **M**, which means the possible values for **M** are **0** or **5**.

However, since **O** and **M** represent different digits, **M = 5**.

In this case, we have a carryover of **1** into the third column, meaning:

$$3J + 1 = I$$

Considering that **J ≠ 0**, the possible solutions are:

$$I = 4,\ J = 1 or I = 7,\ J = 2.$$

– If **O = 5**, then **3O = 15**, producing a carryover of **1** into the middle column.

This requires:

$$3M + 1 = M + 10\ or\ 3M + 1 = M + 20\ or\ 3M + 1 = 2M + 9\ or\ 3M + 1 = 2M + 19.$$

It is clear that no valid integer value for **M** satisfies these cases.

Hence, the only possible solutions are:

$$JMO = 150,\ IMO = 450\ \ and\ \ JMO = 250,\ IMO = 750.$$

Multi-Variable Linear Equations:

Exercises:

(1)

$$\frac{a_1}{a_2} = \frac{1}{2} = \frac{b_1}{b_2} = \frac{5}{10} = \frac{c_1}{c_2} = \frac{3}{6}$$

The system has **infinitely many solutions**.

(2)

$$(a)\ x = 10\ , y = 3$$

$$(b)\ x = 4\ , y = 1$$

$$(c)\ x = 8\ , y = 6$$

$$(d)\ x = 5\ , y = 4$$

(3)

Let:

$$x + y = 42$$
$$x - y = 8$$

By adding the two equations:

$$2x = 50 \Rightarrow x = 25$$

By substituting in the first equation:

$$25 + y = 42 \Rightarrow y = 17$$

So the two numbers are 25 □ 17

(4)

We are told that $a = 6b$.

To convert our other information into an equation, we note that when it is b minutes after , 3 o'clock, it is $b + 60$ minutes after 2 o'clock.

We are told that this time is 15 minutes later than when the time was a minutes after 2 o'clock.

So, we have:

$$\begin{cases} a = 6b \\ a + 15 = b + 60 \end{cases}$$

$\Longrightarrow b = 9.$

Therefore, it was 3 : 09 when she looked at her watch for the second time.

Challenge Problems:

(1)

By adding the two equations:

$$\frac{2}{a} = 20 \Longrightarrow \frac{1}{a} = 10 \Longrightarrow a = \frac{1}{10}$$

By substituting in the second equation:

$$10 + \frac{1}{b} = 14 \Longrightarrow \frac{1}{b} = 4 \Longrightarrow b = \frac{1}{4}$$

(2)

By adding the three equations, we obtain:

$$2(x + y + z) = 18 \Longrightarrow x + y + z = 9$$

Substituting into the first equation gives $z = 1$,

From the second, we get $x = 5$ **, and from the third**, we obtain $y = 3$.

(3)

For example, $\frac{1}{2}, \frac{1}{3}$

To find a general form Let the two fractions be $\frac{1}{a}, \frac{1}{b}$,

Then we have

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{a} \cdot \frac{1}{b} \Longrightarrow \frac{b-a}{ab} = \frac{1}{ab}$$

which gives:

$$b - a = 1 \Longrightarrow b = a + 1$$

Thus, the two fractions are, in general

$$\frac{1}{a}, \frac{1}{a+1}$$

(4)

Let the width and length of the shaded rectangle be x and $3y$, respectively, and let the length of the unshaded rectangle be l.

Since each rectangle has the same perimeter, we have:

$$2(l + y) = 2(x + 3y)$$

Thus,

$$l + y = x + 3y$$

and therefore,

$$l = 3y - 2x.$$

Since the original shape is a square, we also have:

$$x + l = 3y.$$

Substituting gives:

$$x + 3y - 2x = 3y$$

which implies:

$$y = 3x.$$

Hence, the area of the shaded rectangle is:

$$3xy.$$

And the area of the unshaded rectangle is:

$$ly = y(x + 2y) = y(x + 6x) = 7xy.$$

Therefore, the ratio of the area of the shaded rectangle to the unshaded one is:

$$3xy: 7xy = 3: 7.$$

(5)

Let the number of children be n, and let the total amount of money be k halalas.

We have

$$60n = k - 210$$

$$\Longrightarrow k = 60n + 210.$$

From another side,

$$\frac{k+20}{n} = 70$$

$$\Longrightarrow k = 70n - 20.$$

$$\therefore 60n + 210 = 70n - 20,$$

and therefore

$$n = 23,$$

which is the number of children.

(6)

$$3600 = 2^4 \times 3^2 \times 5^2 = 2^a \times 3^b \times 4^c \times 5^d$$

$$\Longrightarrow b = 2\,, d = 2\,,\ 2^4 = 2^a \times 4^c$$

$$\because 4^c = 2^{2c}$$

$$\therefore\ 2^4 = 2^a \times 2^{2c} \Longrightarrow 4 = a + 2c \Longrightarrow a = 4 - 2c$$

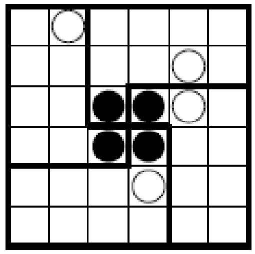
But we are also given that: $a + b + c + d = 7$

$$\Longrightarrow 4 - 2c + 2 + c + 2 = 7 \Longrightarrow c = 1$$

(7)

We want to have as many 5s as possible on the left. We will start by removing 1234 and leaving 5. Then we repeat the process: remove 1234 and leave 5. Unfortunately, we cannot make another 5 on the left. We are only allowed to remove the next 12. Therefore, the largest number we can obtain is: 553451234512345.

(8)



(9)

The only child who could be in kindergarten is the 5-year-old, and because one of the children in kindergarten is a girl, the youngest child is a girl. Muhammad is not the youngest, and because he is younger than Najah, he is not the oldest either. Therefore, Muhammad is either **8 or 13** years old.

- Assuming Muhammad is 8 years old, then Raja is 13 or 15, and the possible pairs representing the ages of Nour and Raja are:

$$(5,13), (5,15), (13,15)$$

The only pair that satisfies the condition that the sum of their ages is divisible by 3 is when Nour is 5 years old and Raja is 13 years old.

- Assuming that Muhammad is 13 years old, then Raja is 15 years old, and the possible pairs representing the ages of Nour and Raja are:

$$(5,15), (8,15)$$

There is no possibility that the sum of the two ages can be divided by three.

Therefore, the conditions of the problem are satisfied **only when Nour is five years old**, which means that Nour must be a **girl**.

(10)

If everyone on the island always told the truth, we would have exactly 100 “yes” answers.

Each liar gives **three** “yes” answers instead of just one, which increases the total number of “yes” answers by 2 for each liar.

Since the total number of “yes” answers given was:

$25 + 25 + 45 + 55 = 150,$

there are 50 **extra** “yes” answers.

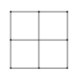
Therefore, the number of liars is:

$50 \div 2 = 25.$

(11)

The question is not easy to solve, and the goal is to encourage the student’s creativity

If we look at any 2×2 square divided into 4 small squares inside the table.



Let's call it a “block.” We will find that any two small squares in it will share a corner or side.

Therefore, we cannot color more than one small square black inside any block in the table.

Now we ask the question: What is the largest number of separate (i.e., non-overlapping) blocks into which we can divide the given table? The answer is 16 blocks!

Therefore, the largest number of small squares that can be colored black under the conditions of the question is 16, and we cannot color 17 small squares in this way.

Percent:

Exercises:

(1)

We have:

$$\frac{7}{x+y}=\frac{k}{x+z}\Longrightarrow k(x+y)=7(x+z)\longrightarrow(1)$$

In the same way

$$\frac{k}{x+z}=\frac{11}{z-y}\Longrightarrow k(z-y)=11(x+z)\longrightarrow(2)$$

By adding the two equations,

$$k(x+y)+k(z-y)=18(x+z)$$

$$\Rightarrow kx+ky+kz-Ky=18(x+z)$$

$$\Rightarrow k(x+z)=18(x+z)\Longrightarrow k=18$$

(2)

Since:

$$\frac{b+2}{2}=\frac{8}{c+3}\Longrightarrow(b+2)(c+3)=16$$

We have the following possible factor pairs:

- $(b+2)(c+3)=1\times 16\Longrightarrow b=-2\,,c=13$

  This is rejected because b is a natural number
- $(b+2)(c+3)=2\times 8\Longrightarrow b=0\,,c=5$

  This is rejected because b is a natural number
- $(b+2)(c+3)=4\times 4\Longrightarrow b=2\,,c=1$

  This case is possible
- $(b+2)(c+3)=8\times 2\Longrightarrow b=6\,,c=-1$

  This is rejected because c is a natural number
- $(b+2)(c+3)=16\times 1\Longrightarrow b=14\,,c=-2$

  This is rejected because c is a natural number

Thus, the only valid solution is $b=2\,,c=1\Longrightarrow a=5$

(3)

Let the number be x. Then

$$\frac{120}{100}x = 36 \Rightarrow x = 36 \times \frac{100}{120} = 30$$

(4)

$$\frac{2890 - 2023}{2890} \times 100 = \frac{867}{2890} \times 100 = 30$$

(5)

She gave her teacher 40%, leaving her with 60%

$$60 \times \frac{60}{100} = 36$$

She then gave her friend **one quarter** of what remained, meaning she kept **three quarters** of the remainder

$$36 \times \frac{3}{4} = 27$$

She then ate **one third** of what was left, so she kept **two thirds** of it

$$27 \times \frac{2}{3} = 18$$

(6)

The sum of the interior angles of a pentagon is:

$$(5 - 2) \times 180 = 540^\circ$$

And the sum of the ratio parts is

$$2 + 3 + 4 + 5 + 6 = 20$$

Thus, the value of one part is

$$\frac{540}{20} = 27^\circ$$

The largest angle corresponds to 6 parts, so

$$6 \times 27 = 162^\circ$$

(7)

$$b = \frac{105}{100}a \quad , \quad b = \frac{85}{100}c$$

$$\Longrightarrow \frac{105}{100}a = \frac{85}{100}c$$

$$\Longrightarrow \frac{a}{c} = \frac{85}{100} \times \frac{100}{105} = \frac{85}{105} = \frac{17}{21}$$

(8)

Let the original length be L and the original width be W.

Thus, the original area is $L \times W$

After the increase:

The new dimensions become:

$$\frac{150}{100}L \quad , \quad \frac{120}{100}W$$

The new area is

$$\frac{150}{100}L \times \frac{120}{100}W = \frac{180}{100}LW$$

Therefore, the percentage increase in area is

$$\frac{\frac{180}{100}LW - LW}{LW} \times 100 = 80\%$$

(9)

Every day, 20% is sold, leaving 80% of the fish remaining each day. Therefore:

After one day, $\frac{80}{100}$ of the quantity remains.

After two days, $\frac{80}{100} \times \frac{80}{100} = \frac{64}{100}$ of the initial amount

Because after two days (i.e., on Tuesday) **2,000 fish** remain, we set:

$$\frac{64}{100}x = 2000 \Longrightarrow x = 2000 \times \frac{100}{64} = 3125$$

(10)

Since:

$$t = \frac{u}{4} \Longrightarrow 4t = u$$

Therefore

$$\frac{4t}{2u} = \frac{u}{2u} = \frac{1}{2}$$

$$\Longrightarrow 4t : 2u = 1 : 2$$

(11)

Since:

$$\frac{x}{yz} : \frac{y}{zx} = 1 : k \Longrightarrow \frac{k}{1} = \frac{y}{zx} \times \frac{yz}{x} = \frac{y^2}{x^2}$$

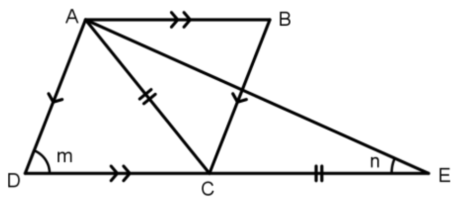
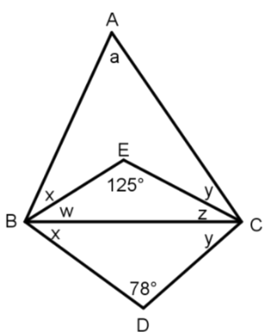
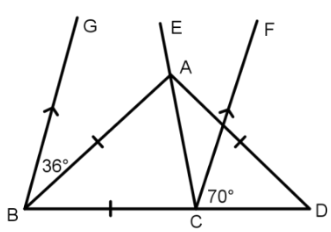
And

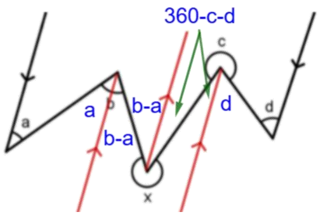
$$yz : zx = 1 : 2 \Longrightarrow \frac{yz}{zx} = \frac{y}{x} = \frac{1}{2}$$

Therefore

$$k = \frac{y^2}{x^2} = \left(\frac{y}{x}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

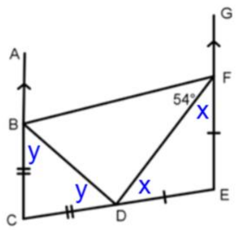
Solutions of Revision Exercises:

(1)	
	$CA = CE \Longrightarrow \angle EAC = \angle AEC = n \Longrightarrow \angle ACD = n + n = 2nAD = CD \Longrightarrow \angle DAC = \angle DCA = 2nD + \angle DAC + \angle DCA = 180 \Longrightarrow 4n + m = 180$
(2)	
	$x + y = 180 - 78 = 102w + z = 180 - 125 = 55A = 180 - (x + y + w + z) = 180 - (102 + 55) = 23°$
(3)	
	$\angle DCF = \angle CBG = 70 \Longrightarrow \angle ABC = 70 - 36 = 34° \Longrightarrow \angle ADC = \angle ABC = 34, \angle BAC = \angle BCA = \frac{180-34}{2} = 73° \angle BAD = 180 - (34 + 34) = 112° \Longrightarrow \angle DAC = 112 - 73 = 39 \Longrightarrow \angle EAD = 180 - 39 = 141°$
(4)	



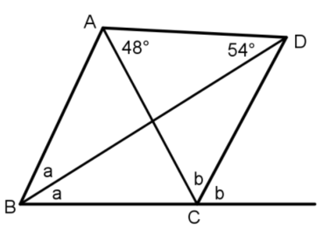
$$x + b - a + 360 - c - d = 360 \Longrightarrow x = c + d + a - b$$

(5)



$$2x + \angle FED = 180\ 2y + \angle BCD = 180 \quad \angle FED + \angle BCD = 180 \Longrightarrow 2x + 2y = 180 \Longrightarrow x + y = 90 \Longrightarrow \angle BDF = 180 - (x + y) = 90$$

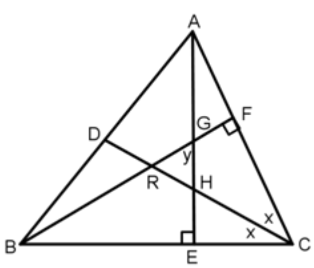
(6)



$$\angle BAC = 2b - 2a\angle BDC = b - a2b = a + 78 \ , \ a = \frac{2}{3}b$$

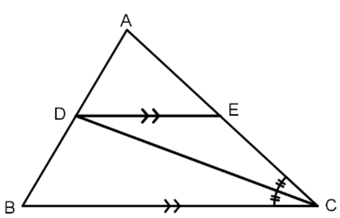
$$\Longrightarrow 2b = \frac{2}{3}b + 78 \Longrightarrow b = 58.5^\circ \ , = a = 39^\circ$$

(7)



$$\angle FRC = 90 - x\angle GHR = \angle EHC = 90 - x\Delta GHR: y + (90 - x) + (90 - x) = 180 \Longrightarrow y = 2x$$

(8)

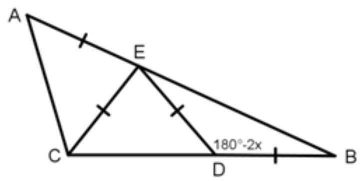


$$\angle ADE = \angle B = 70^\circ,$$

$$\angle EDC = \angle BCD = \frac{40}{2} = 20^\circ$$

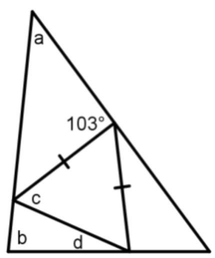
$$\Longrightarrow \angle BDC = 180 - (70 + 20) = 90^\circ$$

(9)



$$\angle DBE = \angle DEB = x \angle DCE = \angle CDE = 2x \angle AEC = \angle DCE + \angle DBE = 3x \angle ACE = \angle CAE = \frac{180 - 3x}{2}$$

(10)



$$\frac{3}{2}a = c \quad , \; 2a = b \;\; , \; \frac{1}{3}a = d77 - a + c = b + d \Longrightarrow 77 - a + \frac{3}{2}a = 2a + \frac{1}{3}a \Longrightarrow 77 \times 6 - 6a + 9a = 12a + 2a \Longrightarrow$$

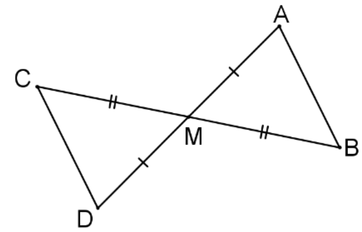
$$77 \times 6 = 11a \;\; \Longrightarrow a = 42°$$

$$b = \frac{42}{2} = 21 \;\; , \; c = \frac{3 \times 42}{2} = 63 \;\; , \; d = \frac{42}{3} = 14$$

Solutions of Triangle Congruence Exercises:

Exercise:

In triangles AMB, DMC, we have:



$$\{MA = MD\ MB = MC\ \angle AMB \cong \angle DMC$$

Therefore, we have AMB, DMC are congruent and that gives the required results:

1) $AB = CD$

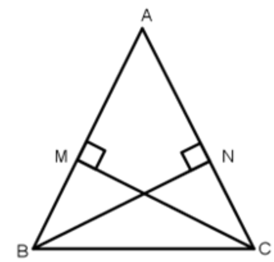
2) $m\angle A = m\angle D$ but they are interior angles. Thus, $AB \parallel CD$.

(1–15)

Ex. No	Congruent Triangles	Congruence Type
1	$ABC \cong NPY$	$A.S.A$
2	None	None
3	$ABC \cong CKA$	$S.S.S$
4	None	None
5	None	None
6	None	None
7	$ABC \cong PQC$	$S.A.S$
8	None	None
9	$ABC \cong AGC$	$A.S.A$
10	$ABC \cong CDA$	$A.S.A$
11	$ABC \cong PST$	$A.S.A$
12	None	None
13	None	None
14	$ABC \cong CDA$	$S.A.S$
15	$ABC \cong MNC$	$A.S.A$

(16)

Notice that:



$m\angle AMC = m\angle ANB = 90°$ and $\angle A$ is a common angle for

both triangles. Thus, $\angle NBA \cong \angle ACM$, but it is given that:

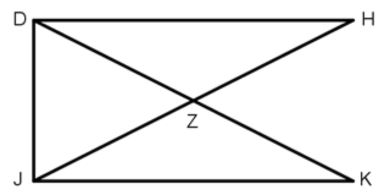
$AB = AC$

Therefore, from $(A.S.A)$ we have,

$$\Delta ABN \cong \Delta ACM$$

(17)

From the given conditions, notice that:



$\Delta JDH, \Delta DJK$

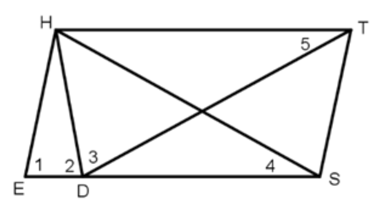
Which gives that:

$$\angle H = \angle K$$

(18)

We will try to find the conditions for the congruence

$$\Delta HES, \Delta HDT$$



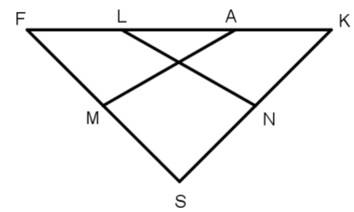
We are given $ES = DT$,$\angle 1 = \angle 3$,

In ΔHED, we have $HD = HD$ since ,$\angle 1 = \angle 2$

Thus $\Delta HES, \Delta HDT$ are congruent and that gives that ,$\angle 4 = \angle 5$.

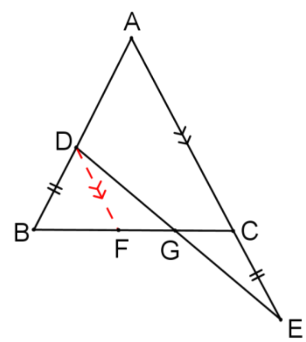
(19)

$$FL = AK \Longrightarrow FL + LA = AK + LA \Longrightarrow FA = KLSF = SK$$
$$\Longrightarrow \frac{1}{2}SF = \frac{1}{2}SK \Longrightarrow MF = NKSF = SK$$
$$\Longrightarrow \angle K = \angle F \Delta FAM \cong KLN\ (SAS) \Longrightarrow AM \cong LN$$



(20)

From point D, we draw $DF \parallel AE$ as shown.



Notice that: $\angle FDG = \angle CEG$, $\angle DGF = \angle EGC$ by parallelism.

But since $\angle BFD = \angle BCA = \angle DBF$. Hence,

$$DF = DB = CE \ \Rightarrow \ \Delta DFG \cong \Delta ECG \ \Rightarrow DG = GE$$

Solutions (Number Theory)

(1)

We know that the number is even if it is divisible by 4 (since it will be divisible by 2 after dividing it by 2 once. But we know from the divisibility rule of 4, the last 2 digits have to be divisible by 4. But 38 is not divisible by 4. Thus, the result would be odd.

(2)

We cannot divided the books, since the sum of 5 odd numbers is odd while 50 is even.

(3)

Since a, b are two consecutive integers. One of them is even while the other is odd. But their multiplication is c. which means that c is even. So the numbers a, b, c have two evens and one odd. And their squares should have the same parity (square of even is even and square of odd is odd). Thus, $N = even + even + odd = odd$.

(4)

The pattern started with 1,2, if we continue the pattern we will noticed that the parity of the numbers follows the pattern $(O, E, O, O, E, O, O, E, O, \dots)$. To decide the parity of the 2003^{th} term, we divide 2003 by 3 to get a remainder of 2. Thus, the numbers will be divided as O, E, O many times and then we will be left with O, E at the end (since the remainder is 2). So, the 2003^{th} will be even.

(5)

Let $k = (x + 1)(y + 2)(z + 3)$, then we have three cases:

Case 1: x, y are odd, then $x + 1$ is even. Thus, k will be even.

Case 2: x, z are odd, then $x + 1$ is even. Thus, k will be even.

Case 1: y, z are odd, then $z + 3$ is even. Thus, k will be even.

(6)

It is clear that the number of odd numbers in the list is odd (exactly 1009 odd numbers). So if we add all the numbers the result would be odd. But since addition and subtraction does not affect the parity (only thing that affects the parity is the number of odd numbers). The final result must be odd.

(7)

(a) Yes, for example $1+2-3+4+5+6-7-8$.

(b) Not possible, since the sum $1+2+\cdots+9=45$ is odd. And as we mentioned before that subtraction or addition does not affect the parity. It means that the final result would be odd while 0 is even.

(8)

Not possible.

Let the number of wins be x, draws be y, and losses be $20-x-y$. Then the total score is:

$$3x+y-(20-x-y)=2x+2y-20,$$

which is an even number and therefore cannot equal 39.

Another method.

Not possible, Since we have an addition/subtraction of 20 odd numbers (the number of odd numbers is even). Thus, the final score can only be even, but 39 is odd. So it is not possible.

(9)

(a) It is possible, we put the oranges in the basket in the following way 1,2,1,2, ... ,1,2.

(b) Not possible, this is clear by looking at the parities of the oranges in consecutive baskets. They have to be:

$$O, E, O, E, \ldots$$

But that means the first and last basket will have the same parity and so they cannot differ by 1.

(10)

(a), (b) Not possible, the worm can only jump an even integer number of steps. (c) is possible, that is by jumping 6 cm twice to the right, followed by 8 cm once to the left.

(11)

This is not an easy question; it seems possible but after many tries it would look impossible. But how do we prove that it is indeed impossible?

One way to tackle this problem is to look at the position of each of the numbers on the line (first number has position 1 while the last number has position 10). Then we can construct the following equations:

First, the sum of positions is $1 + 2 + \cdots + 10 = 55$. But looking at it from another point of view.

The sum of positions of 1,1 is $a + (a + 2) = 2a + 2$. (If we assume that a is the position of the first 1)

The sum of positions of 2,2 is $b + (b + 3) = 2b + 3$.

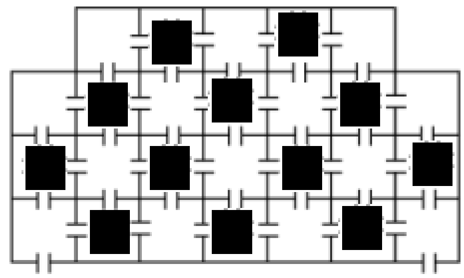
The sum of positions of 3,3 is $c + (c + 4) = 2c + 4$.

The sum of positions of 4,4 is $d + (d + 5) = 2d + 5$.

The sum of positions of 5,5 is $e + (e + 26) = 2e + 6$.

Note that the sum of the positions above is EVEN (since we only have two odd). However, 55 is odd. This shows that it is impossible to arrange the numbers as required by the problem statement.

(12)



This is a similar question to the previous one. It seems possible but after many tries we can see that it hard to show a way to do it. So, how can we show that it is impossible?

Notice that if we color the rooms as in the figure above. It means that we go from black to whit rooms and vice versa. We start at a white room and want to end at another white room passing through all the rooms. This means that the number of black and white rooms should be equal. But this is not the case. So it is impossible!

This question may seem unrelated to the topic. However, if you look closely, we are pairing the white and black rooms together (so it is not that the parity of black and white rooms has to be equal, but that the **black** room has to be paired with a **white** room, and that is impossible as shown above)

Solutions (Combinations)

The Two Fundamental Counting Principles:

(1)

$$6 \times 4 = 24$$

(2)

$$9 \times 10 \times 10 = 900$$

(3)

$$8 \times 7 \times 6 \times 5 \times 4 = 6720$$

(4)

$$9 \times 10 = 90$$

(5)

$$4 \times 3 \times 2 \times 1 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 17280$$

(6)

$$7 + 10 = 17$$

(7)

$$3 \times 5 + 4 \times 5 + 4 \times 3 = 47$$

(8)

$$4 \times 3 \times 9 \times 8 \times 7 = 6048$$

(9)

$$8 \times 20 + 13 \times 20 + 8 \times 13 = 524$$

(10)

a. $9 \times 5 = 45$
b. $3 \times 5 = 15$
c. $2 \times 4 = 8$
d. $4 \times 3 + 5 \times 2 = 22$

Number of Numbers and Strings:

(11)

$$9 \times 9 \times 8 \times 7 = 4536$$

(12)

$$9 \times 10 \times 10 \times 10 \times 5 = 45000$$

(13)

$$(4 \times 4) + (3 \times 4) = 16 + 12 = 28$$

(14)

$$3 \times 9 \times 9 = 243$$

(15)

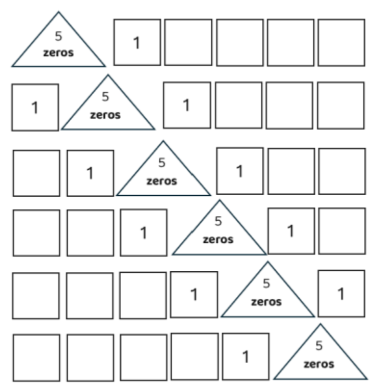
Divisible by 3: (1,8,3), (1,8,9), (7,8,9), (7,8,3),

and each has six arrangements:

$$4 \times 6 = 24$$

(16)

The five consecutive zeros can start in any position from 1 to 6, with the condition that they are not adjacent to any other zero.



If they are at the edge (starting at 1 or 6), then there are 4 free slots: 2^4 for each.

If they are in the middle (starting at 2–5), then there are 3 free slots: 2^3 for each case.

Total: $2^3 \cdot 4 + 2^4 \cdot 2 = 64$

(17)

$$5 \times 10^3 = 5000$$

(18)

$$(4 \times 5^5) + (5^6) = 28125$$

(19)

We choose the position of "1" (4 ways) and "3" (3 ways), and the remaining digits have 8 possibilities each, then subtract those that start with zero: $4 \times 3 \times 8 \times 8 - 3 \times 2 \times 8 = 720.$

(20)

$$900 - 5^3 = 775$$

(21)

We forbid two even digits from being adjacent, and the first digit cannot be zero.
The number of choices for odd digits is always 5, and for even digits, it's 4 in the first (leftmost) position and 5 in the others. Allowed patterns: OOOO, OEOO, OOEO, OOOE, OEOE (each has 5^4) and EOOO, EOEO, EOOE (each has 4×5^3).

So: $5 \times 625 + 3 \times (4 \times 125) = 3125 + 1500 = 4625$ numbers.

(22)

$9 \times 9 \times 9 = 729$

(23)

We choose the ordered pair (first, last) such that |first − last| = 2, with zero not allowed in the first position. For first digits from 1 to 9, the number of valid pairs is 15 (1 gives 1 pair, 2 gives 2, 3–7 each give 2, 8 gives 1, 9 gives 1). After fixing the first and last digits, the three middle digits are chosen from the remaining eight distinct digits, in order: $8 \times 7 \times 6 = 336$. Therefore, the total number$= 15 \times 336 = 5040$.

(24)

In the ones place 10, and in the tens place 10 → total $= 20$.

(25)

(a) $8^3 = 512$
(b) $8^4 = 4096$
(c) $9^2 \times 8 = 648$

Number of Words:

(26)

$$26 \times 25 \times 24 \times 23 \times 22 = 7{,}893{,}600$$

(27)

$$7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$

(28)

$$(26 \times 25 \times 24) \times 26^3 = 274{,}185{,}600$$

(29)

$$3 \times 25 \times 25 = 1875$$

(30)

Either M appears once $(3 \times 25 \times 24)$, twice (3×25), or three times (1). Adding them up: $1800 + 75 + 1 = 1876$.

(31)

$$26^3 - 25^3 = 1951$$

(32)

$$3^1 + 3^2 + 3^3 + 3^4 = 120$$

(33)

We start with one letter from A, B, or C (3 ways). Each letter can be followed by only two allowed letters (A or C after A, A or B after B, 9 or C after C). Thus, the number of words doubles at each step. After 7 letters, the total $= 3 \times 2^6 = 192$ good words.

(34)

$$21 \times 26^3 = 369{,}096$$

(35)

If there are five letters between x and y, the distance between them is 7 positions. So, the pair can start in any of the positions 1 to 19 $(26 - 7 + 1 = 20)$. For each position, there are 2 possible orders (x before y or y before x). Then, the remaining 24 letters can be arranged in any order: 24!

Therefore, the total number $=$

$$20 \times 2 \times 24 \times 23 \times 22 ... \times 2 \times 1$$