وزارة التعليم
Ministry of Education

موهبة
Mawhiba

نسمو
NSMO

National Science and Mathematics Olympiad

Learning Materials for the Mathematic 01 Track

National Teams Competition 2026

Mathematic

بسم الله الرحمن الرحيم

NSMO نسمو

موهبة Mawhiba | وزارة التعليم Ministry of Education

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Introduction

Our exceptional sons and daughters,

We are delighted to congratulate you on successfully completing the **Cities and Governorates stage** and qualifying for the **General Administrations stage**—an important, advanced step on your path toward mathematical challenge and innovation.

This **resource packet** is designed to **expand your understanding** across the four main branches of mathematics: **Combinatorics, Geometry, Algebra, and Number Theory**. We will focus on advanced concepts in **permutations, similarity of polygons and proportional segments, factorization, and prime and composite numbers**.

This stage aims to **refine your skills in analytical thinking** and help you **connect mathematical concepts** to one another, applying them effectively in various problem situations.

This resource packet offers a valuable opportunity to deepen your understanding of mathematical patterns and to apply logical reasoning for justification and problem-solving using organized methods.

We are confident in your abilities and look forward to seeing you **excel** in this crucial phase of your journey toward excellence.

The Scientific Team for the National Science and Mathematics Olympiad (NSMO) – Mathematics Track

First Unit: ALGEBRA

One-Variable Linear Equations

To solve a one-variable linear equation, follow the steps below in order:

- **Eliminating Denominators (if any)**

If there are fractions in the equation, multiply each term by the least common multiple **(LCM)** of the denominators to eliminate them.

- **Expand parentheses**

Use the distributive property to simplify parentheses, such as:

$$2(x+3)=2x+6$$

- **Move terms**

Move all the terms with the variable to one side and all the constants to the other using addition and subtraction.

- **Combine Like Terms**

Simplify both sides of the equation so that it takes the form:

$$ax=b \ \ , where\ a\ and\ b\ are\ constants.$$

- **Dividing by the variable's coefficient**

We divide both sides by a (the variable coefficient) to get the solution:

$$ax=b \Longrightarrow x=\frac{b}{a}$$

Note that:

- When $a \neq 0$ we have a unique solution:

$$x=\frac{b}{a}$$

- When $a=0\ ,b\neq 0$ there is no solution.
- When $a=0\ ,b=0$ any real number is a solution to the equation.

- **Check your answer**

We can check our answer by substituting it back into the original equation. If the original equation is not satisfied by our answer, then we made a mistake.

- **Note:** Sometimes we do not follow the exact order of the previous steps when a more direct method is available.

Example:

Solve:

$$\frac{1}{10}\left\{\frac{1}{9}\left[\frac{1}{5}\left(\frac{x+2}{3}+8\right)+16\right]+8\right\}=1$$

Solution:

$$\frac{1}{10}\left\{\frac{1}{9}\left[\frac{1}{5}\left(\frac{x+2}{3}+8\right)+16\right]+8\right\}=1$$

$$\frac{1}{9}\left[\frac{1}{5}\left(\frac{x+2}{3}+8\right)+16\right]+8=10$$

$$\frac{1}{9}\left[\frac{1}{5}\left(\frac{x+2}{3}+8\right)+16\right]=2$$

$$\frac{1}{5}\left(\frac{x+2}{3}+8\right)+16=18$$

$$\frac{1}{5}\left(\frac{x+2}{3}+8\right)=2$$

$$\frac{x+2}{3}+8=10$$

$$\frac{x+2}{3}=2$$

$$x+2=6$$

$$x=4$$

Exercises:

(1) Solve:

$$1-\frac{x-\frac{1+3x}{5}}{3}=\frac{x}{2}-\frac{2x-\frac{10-6x}{7}}{2}$$

(2) If a, b, c are fixed positive numbers, solve the equation:

$$\frac{x-a-b}{c}+\frac{x-b-c}{a}+\frac{x-c-a}{b}=3$$

(3) Solve:

$$(x-3)^2+(x+1)^2+(4x-5)^2=0$$

(4) Solve:

$$ax+b-\frac{5x+2ab}{5}=\frac{1}{4}$$

(5) Given that the equation

$$a(2x+3)+3bx=12x+5$$

has infinitely many solutions for x. Find the values of $a$and b.

(6) Given that the equation

$$2a(x+6)=4x+1$$

has no solution, where a is a parameter, find the value of a

(7) Given that the equation

$$kx=12$$

has positive integer solution only, where k is an integer. Find the number of possible values of k.

(8) How many possible positive integer values of x satisfy the equation

$$\frac{1}{x}+\frac{1}{x+1}+\frac{1}{x+2}=\frac{13}{12}$$

(9) Given that the solution of equation

$$3a-x=\frac{x}{2}+3$$

is 4. Find the value of $(-a)^2-2a$

(10) Solve the equation

$$\frac{x-n}{m}-\frac{x-m}{n}=\frac{m}{n}$$

(where $mn \neq 0$).

Multi-Variable Linear Equations

The general form of a system of two linear equations in two variables is:

$$\begin{cases} a_1x + b_1y = c_1 & (1) \\ a_2x + b_2y = c_2 & (2) \end{cases}$$

Where $a_1, b_1, c_1, a_2, b_2, c_2$ are real numbers

We have two linear equations in two variables, and they represent two straight lines in the coordinate plane.

The solution of the system corresponds to the point(s) of intersection of these two lines (since such a point satisfies both equations).

To eliminate one of the variables and solve the system, we use:

(i) the usual algebraic operations.

(ii) the substitution method.

In many cases, method (i) is more efficient.

When

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

- Thus, the two lines intersect at exactly one point, and therefore the system has a unique solution, which is

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \quad ، \quad y = \frac{a_1c_2 - a_2b_1}{a_1b_2 - a_2b_1}$$

When

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

- Then the two lines coincide, and therefore all their points are common. As a result, the system has infinitely many solutions

When

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

- Then the two lines are parallel and have no points of intersection; therefore, the system has no solution.

Exercises:

(1) Solve the following system of equations:

$$\begin{cases} \dfrac{x-y}{5} - \dfrac{x+y}{4} = \dfrac{1}{2}, \\ 2(x-y) - 3(x+y) + 1 = 0 \end{cases}$$

(2) Solve the following system of equations:

$$\begin{cases} 5.4x + 4.6y = 104 \\ 4.6x + 5.4y = 96 \end{cases}$$

(3) Solve the following system of equations:

$$\begin{cases} (x+2)(5x+y) = 16 \\ 5x + y = 7 \end{cases}$$

(4) Solve the following system of equations:

$$\begin{cases} \dfrac{x}{2} = \dfrac{y}{3} = \dfrac{z}{5} \\ x + 3y + 6z = 15 \end{cases}$$

(5) Solve the following system of equations:

$$\begin{cases} x + 2y = 5 \\ y + 2z = 8 \\ z + 2u = 11 \\ u + 2x = 6 \end{cases}$$

(6) Solve the following system of equations:

$$\begin{cases} 5x - y + 3 = a \\ 5y - z + 3x = b \\ 5z - z + 3y = c \end{cases}$$

(7) Given that $x = 2, \ \ y = 1$ is the solution of system

$$\begin{cases} ax + by = 7, \\ bx + cy = 5 \end{cases}$$

What is the relation between a, c ?

(8) Find the value of (a, b, c) if both systems have the same solution

$$\begin{cases} 3x - y = 5 \\ 2x + y - z = 0 \\ 4ax + 5by - z = -22 \end{cases} \quad , \quad \begin{cases} ax - by + z = 8 \\ x + y + 5 = c \\ 2x + 3y = -4 \end{cases}$$

(9) Determine the values of k such that the system of equations

$$\begin{cases} kx - y = -\frac{1}{3} \\ 3y = 1 - 6x \end{cases}$$

$a)$ has unique solution $\quad$ $b)$ no solution $\quad$ $c)$ infinitely many solutions

(10) Given that x, y, z satisfy the system of equations

$$\begin{cases} 2010(x - y) + 2011(y - z) + 2012(z - x) = 0 \\ \\ 2010^2(x - y) + 2011^2(y - z) + 2012^2(z - x) = 2011 \end{cases}$$

find the value of $z - y$

(11) Solve the following system of equations:

$$\begin{cases} x - y - z = 5 \\ y - z - x = 1 \\ z - x - y = -15 \end{cases}$$

(12) Solve the following system of equations:

$$\begin{cases} x - y + z = 1 \\ y - z + u = 2 \\ z - u + v = 3 \\ u - v + x = 4 \\ v - x + y = 5 \end{cases}$$

Factoring

Factoring algebraic expressions means rewriting an expression as a product of simpler expressions, called factors.

<table>
<tr><th colspan="2">Number of Terms in the Expression: Binomial Expression.
(After factoring out the common factor, the expression becomes)</th></tr>
<tr><td>Difference of two squares</td><td>$a^2 - b^2 = (a+b)(a-b)$</td></tr>
<tr><td>Difference of two cubes</td><td>$a^3 - b^3 = (a-b)(a^2+ab+b^2)$</td></tr>
<tr><td>Sum of two cubes</td><td>$a^3 + b^3 = (a+b)(a^2-ab+b^2)$</td></tr>
</table>

<table>
<tr><th colspan="2">Number of Terms in the Expression: A Trinomial.
$ax^2 + bx + c$ where a, b, c are constants and $a \neq 0$</th></tr>
<tr><td>when $a = 1$</td><td>1) Before starting the factorization process, the trinomial should be written in descending order according to the variable used.
2) We must look for a common factor among the algebraic terms and factor it out first, then proceed to factor the remaining expression.
3) If the sign of the last term (the constant term) is positive, then the signs of its factors will be the same and will follow the sign of the middle term in the original expression.
$x^2 + 5x + 6 = (x+3)(x+2)$
$x^2 - 5x + 6 = (x-3)(x-2)$
4) If the sign of the last term is negative, then the signs of its factors will be different, and we choose the pair whose difference matches the sign of the middle term.
$x^2 - 5x - 6 = (x-6)(x+1)$
$x^2 + 5x - 6 = (x+6)(x-1)$</td></tr>
<tr><td>when $a \neq 0,1$</td><td>The expression is factored into two binomials as follows:
1) Factor the first term ax^2into two factors whose product is ax^2.
2) Factor the last term c into two factors whose product is c.
3) (product of the outer terms + product of the inner terms) = the middle term bx.</td></tr>
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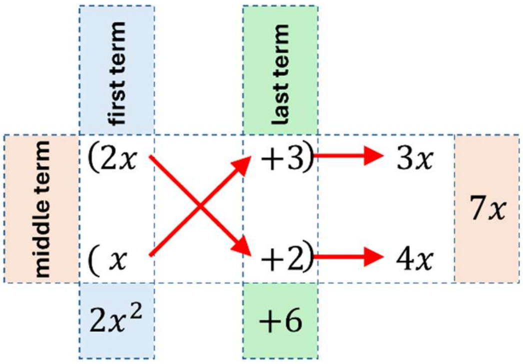
Example 1:

Factor

$$2x^2 + 7x + 6$$

Solution:



first term $2x \cdot x = 2x^2$

last term $2 \cdot 3 = 6$

middle term $3x + 4x = 7x$

Therefore

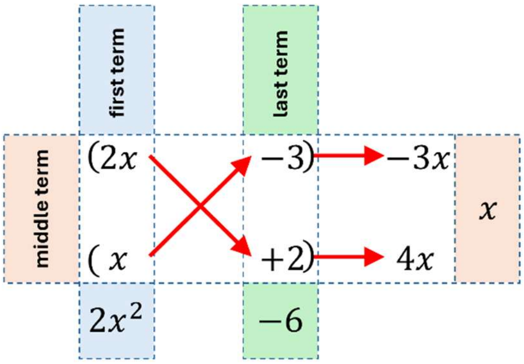
$$2x^2 + 7x + 6 = (2x + 3)(x + 2)$$

Example 2:

Factor

$$2x^2 + x - 6$$

Solution:



first term $2x \cdot x = 2x^2$

last term $2 \cdot (-3) = -6$

middle term $-3x + 4x = x$

Therefore

$$2x^2 + 7x - 6 = (2x - 3)(x + 2)$$

Exercises:

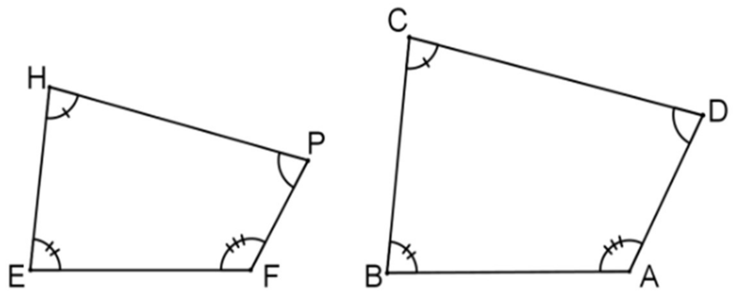
Factor:

1) $x^2 - 8x + 7$	2) $x^2 - 6x - 7$
3) $x^2 - 25$	4) $2x^2 - 50$
5) $x^2 - 13x + 42$	6) $2x^2 + 5x + 2$
7) $x^3 - 1000$	8) $15x^3 + 7x^2 - 2x$
9) $5x^3 - 625$	10) $30x^4 + 5x^3 - 5x^2$
11) $x^2 - 2x + 1$	12) $x^2 + 10x + 25$
13) $6x^3y - 13x^2y + 6xy$	14) $2x^2 + 10x + 12$
15) $12x^2y^2 - 15xy^2 - 63y^2$	16) $24x^3 + 10x^2y - 50xy^2$
17) $9 - 4y^2$	18) $x^{12} - 1$
19) $\frac{1}{4}a^4 - \frac{1}{9}$	20) $y^6 - 81$

Second Unit: Geometry

Similarity

Definition 1:



Any two polygons are **similar** if the two following conditions are met:

(1) The corresponding angles are equal in measure.

(2) The lengths of the corresponding sides are proportional

So, in the polygons above, if we have:

$$\left.\begin{cases} \angle A = \angle F \\ \angle B = \angle E \\ \angle C = \angle H \\ \angle D = \angle P \\ \dfrac{AB}{FE} = \dfrac{BC}{EH} = \dfrac{CD}{HP} = \dfrac{DA}{PF} \end{cases}\right\} \Rightarrow ABCD \sim FEHP$$

Theorem 1:

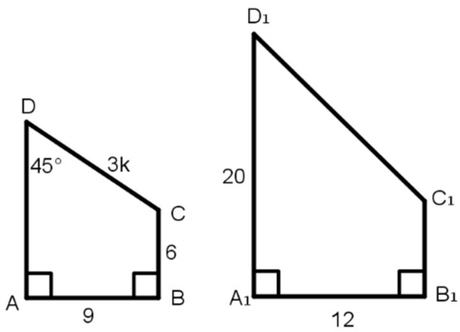
The ratio between the perimeters of similar polygons is equal to the ratio of any two corresponding sides

Exercises:

In exercises $(1-8)$ choose the appropriate answer (only one is correct):

Exercise	Polygons	Cannot be similar	Can be similar	Always similar
(1)	**Equilateral** triangles			
(2)	**Isosceles triangles**			
(3)	**Squares**			
(4)	**Rhombuses**			
(5)	**Right triangles**			
(6)	**Scalene Triangles**			
(7)	**Rectangles**			
(8)	**Right triangle and acute triangle**			

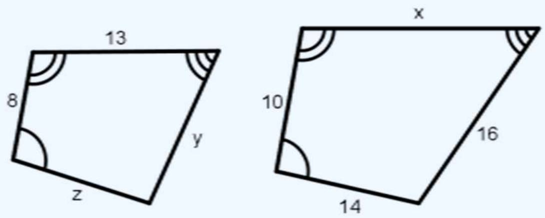
Exercises $(9-16)$ are on the diagrams below. Where quadrilaterals $ABCD \sim A_1B_1C_1D_1$ are similar:



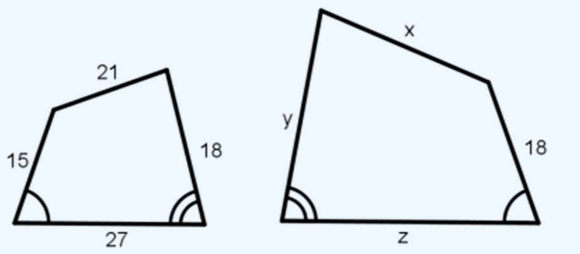
(9)	Similarity ratio of $ABCD$ ، $A_1B_1C_1D_1$.	
(10)	Type of quadrilateral $A_1B_1C_1D_1$	
(11)	Measure of $m\angle D_1$	
(12)	Measure of $m\angle C_1$	
(13)	Measure of C_1B_1	
(14)	Measure of AD	
(15)	Measure of C_1D_1	
(16)	Ratio of the perimeters of the quadrilaterals	

Find the values of x, y, z if the two polygons in each exercise are similar.

(17)



(18)



Triangle Similarity

Theorem (2):
For any two similar triangles we have:

- The corresponding sides are proportional.
- The corresponding altitudes are proportional and have the same ratio as the similarity ratio.
- The corresponding medians are proportional and have the same ratio as the similarity ratio.

Similarity Postulate (AAA)

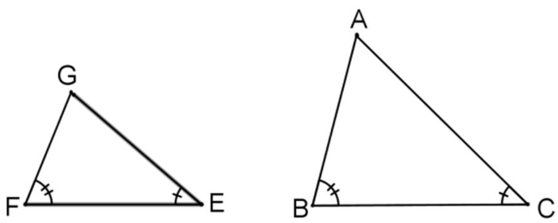
Theorem 3:
Similarity Postulate: AAA
Two triangles are similar if their corresponding angles are equal in measure. We call this case (AAA) (Angle- Angle- Angle). Of course, it suffices to show similarity using two angles instead of all three since the sum of angles in a triangle is 180^o.

In the adjacent triangles $\Delta ABC, \Delta GFE$, if

$$\angle B = \angle F, \angle E = \angle C.$$

Then the triangles are similar and we get that:

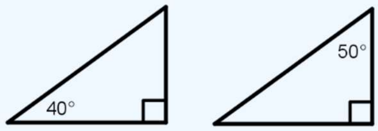
$$\frac{AB}{GF} = \frac{BC}{FE} = \frac{CA}{EG}.$$



Exercises:

In $(1-9)$, determine the similar triangles or state that there aren't any if you cannot find any.

(1)



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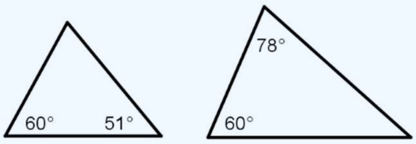
(2)



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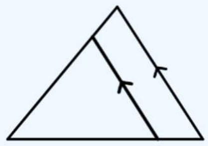
(3)



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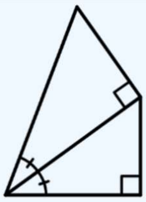
(4)



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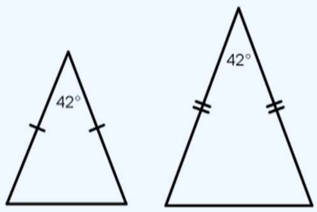
(5)



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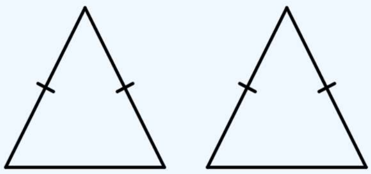
(6)



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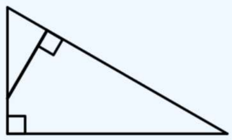

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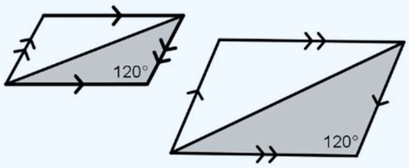
(8)



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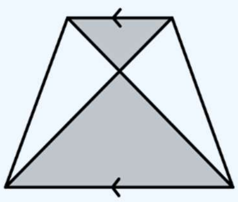
(9)



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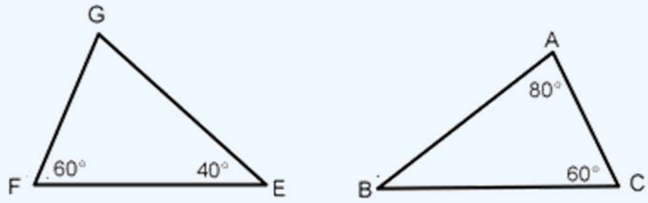
(10) Look at the adjacent figure, and answer the following:

$\angle ABC =$ ________

$\Delta ABC \sim$ ________

$\frac{AB}{.....} = \frac{BC}{.....} = \frac{AC}{.....}$

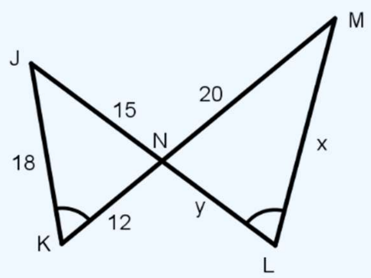


(11) Look at the adjacent figure, and answer the following:

$(a)\ \Delta JKN \sim$ … … … … … ..

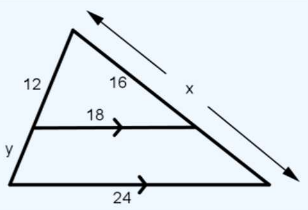
(b) $\frac{15}{.....} = \frac{12}{.....}$, $\frac{15}{.....} = \frac{18}{.....}$

$(c)\ x = \cdots$, $y = \cdots$



Find the values of x, y in the following exercises:

(14)



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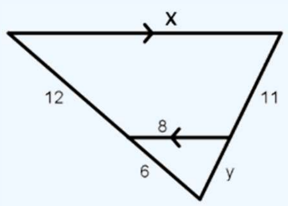
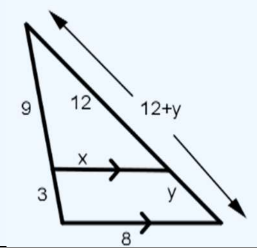


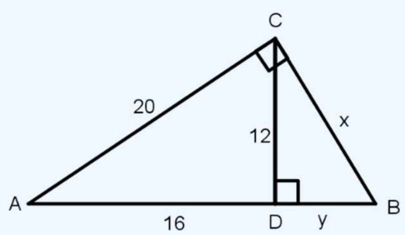
(12)



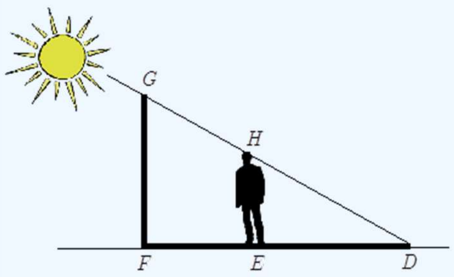
(15) In the adjacent figure:

a) Give two triangles that are similar to ABC.

b) Find the value of x, y.



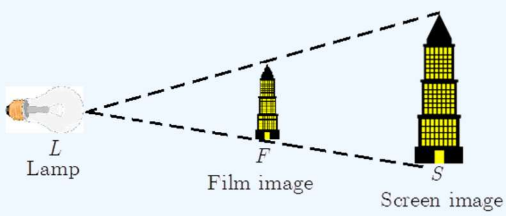
(16) To approximate the height of the basketball stand, one of the players whose height is exactly $2m$ stood so that his shadow ends with the end of stand's shadow. We found that:



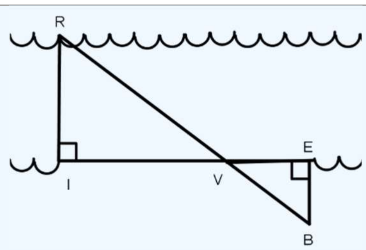
$\underline{DE} = 1.6,\ \underline{DF} = 4.4$

Find the height of the stand.

(17) In the adjacent figure, a picture of a building appears on the film as well as the screen (light source projects the film image on the screen). If $LF = 6cm, LS = 24m$, and the building height on the screen was $2.2m$. What is the height of the building on the film?



(18) In the adjacent figure, If $IV = 63m, VE = 20m, BE = 15m$, Find the width of the river.



Similarity Postulates $(SAS), (SSS)$

Theorem 4:

Two triangles are similar if a corresponding angle is equal, and the ratio of the two corresponding sides that form the angle are equal. We call this case $(S.A.S)$ (side- angle- side).

Theorem 5:

Two triangles are similar if the ratio of the three corresponding sides are equal. We call this similarity $(S.S.S)$ (side-side-side).

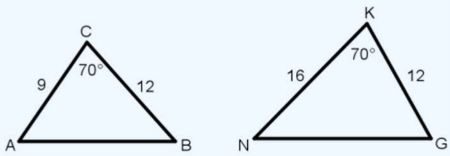
Then, for any two similar triangles we have:

1. Corresponding sides are proportional.
2. Corresponding altitudes are proportional and have the same ratio as the similarity ratio.
3. Corresponding medians are proportional and have the same ratio as the similarity ratio.
4. Corresponding angle bisectors are proportional and have the same ratio as the similarity ratio.
5. The ratio of their perimeters is equal to the similarity ratio.
6. The ratio of their areas is the square of the similarity ratio.

Exercises:

In $(19 - 24)$, determine the similar triangles, and the similarity postulate that you used.

(19)



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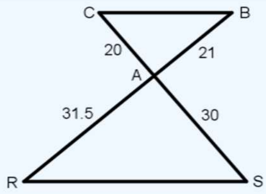

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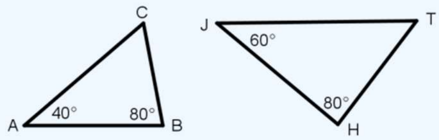
(21)



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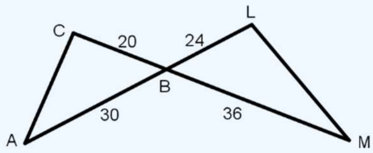

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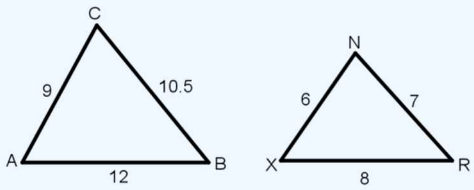

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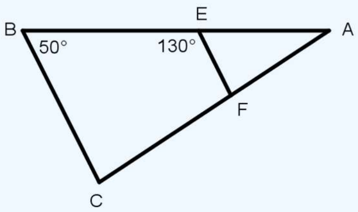

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Proportional Lengths

Theorem 6: (Thales Theorem)

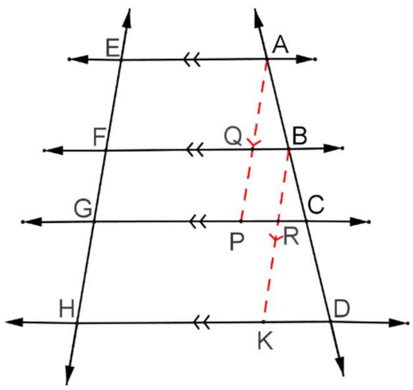
If two lines are intersected by several parallel lines, the segments formed on one transversal are proportional to the segments formed on the other transversal.

Proof:

Here, we have $AE \parallel BF \parallel GC \parallel HD$, and are intersected by lines AD, EH.

We need to show that:

$$\frac{AB}{EF} = \frac{BC}{FG} = \frac{CD}{GH}$$



To prove this theorem, we draw AP and BK parallel to EH as shown in the adjacent figure. Since:

$AQ \parallel FE$ and $QP \parallel GH$, and $BR \parallel FG$ and $RK \parallel GH$ are parallels to sides,

Then, $AQ = EF, BR = QP = FG, RK = GH.$

In $\triangle APC, BQ \parallel PC$, hence:

$$\frac{AB}{BC} = \frac{AQ}{QP}, \frac{AB}{BC} = \frac{EF}{FG} \Rightarrow \frac{AB}{EF} = \frac{BC}{FG}.$$

Similarly, in $\triangle BKD, BQ \parallel PC$, we obtain:

$$\frac{BC}{CD} = \frac{BR}{RK} \Rightarrow \frac{BC}{FG} = \frac{CD}{GH}.$$

Hence:

$$\frac{AB}{EF} = \frac{BC}{FG} = \frac{CD}{GH}.$$

Theorem 7:
If a line is drawn parallel to one side of a triangle and intersects the other two sides, it divides them into segments proportional in length.

Theorem 8: (Angle Bisector Theorem)
If the vertex angle of a triangle (or its external angle) is bisected internally or externally, the ratio of the segments of the base equals the ratio of the lengths of the other two sides.

Proof:

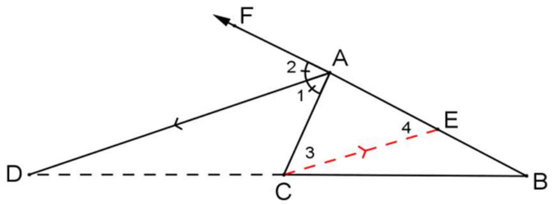


Figure 1

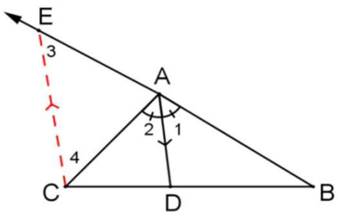


Figure 2

In figure 1: Ray AD Bisects $\angle CAB$ and intersects BC at D.

In figure 2: Ray AD Bisects $\angle CAF$ and intersects BC at D.

We need to prove:

$$\frac{AB}{AC} = \frac{BD}{DC}.$$

To prove this, draw $CE \parallel AD$ intersecting AB at E. Then:

$$\angle 1 = \angle 3 (corresponding), \angle 2 = \angle 4 (alternate).$$

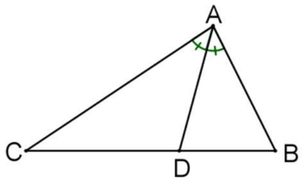
Thus: $\angle 3 = \angle 4$. Therefore $AE \parallel AC$. Since $CE \parallel AD$, we have:

$$\frac{AB}{AE} = \frac{BD}{DC}.$$

Hence:

$$\frac{AB}{AC} = \frac{BD}{DC}.$$

Notes:

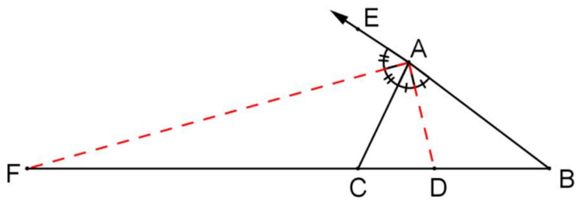


1. The converse of the theorem holds: if D divides BC such that

$$\frac{AB}{AC} = \frac{BD}{DC}$$

then AD bisects $\angle A$.

2. AF and AD are the internal and external bisectors of $\angle A$.



Hence:

$$\frac{AB}{AC} = \frac{BD}{DC}, \frac{AB}{AC} = \frac{BF}{FC}$$

thus

$$\frac{BF}{FC} = \frac{BD}{DC}.$$

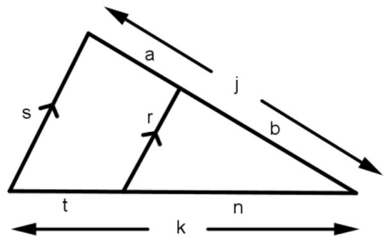
Important Note:

The measure of the angle between the internal and external bisectors of the same angle is $90°$.

Exercises:

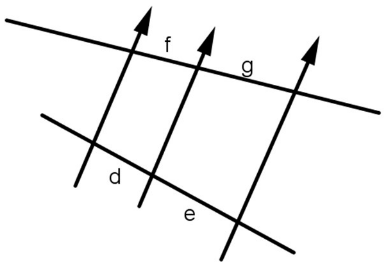
(1) Look at the following figure, and decide which of the relations are correct.

Part	Relation	Answer
(a)	$\frac{r}{s}=\frac{a}{b}$	
(b)	$\frac{t}{k}=\frac{a}{j}$	
(c)	$\frac{j}{a}=\frac{s}{r}$	
(d)	$\frac{r}{s}=\frac{n}{k}$	
(e)	$\frac{a}{b}=\frac{n}{t}$	
(f)	$\frac{b}{j}=\frac{t}{k}$	



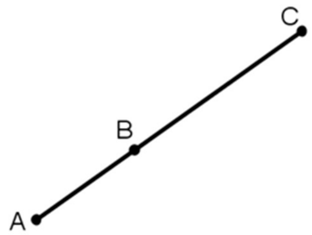
(2) Look at the following figure, and decide which of the ratios are correct.

Part	Relation	Answer
(a)	$\frac{d}{f}=\frac{g}{e}$	
(b)	$\frac{f}{g}=\frac{e}{d}$	
(c)	$\frac{g}{f}=\frac{e}{d}$	
(d)	$\frac{d}{f}=\frac{e}{g}$	



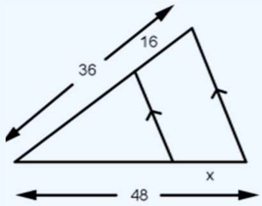
In $(3-6)$, if $\frac{AB}{BC}=\frac{3}{5}$, complete the other entries of the table.

	(3)	(4)	(5)	(6)
AB	6			
BC		25		
AC			56	100



In $(7-15)$, find the values of x, y.

(7)



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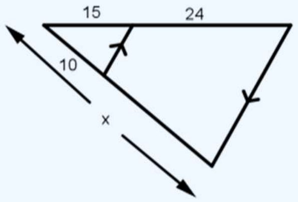

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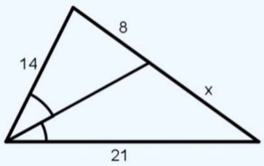

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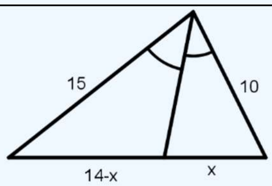

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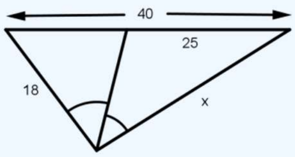

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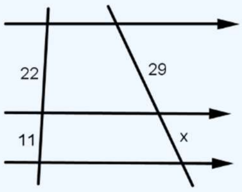

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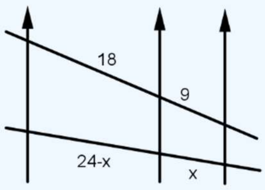

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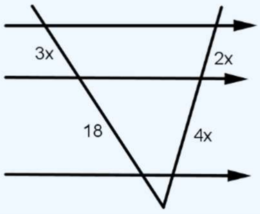


Third Unit: Number Theory

Prime and Composite Numbers

Revision:

- Between all the positive integers, the number 1 is the only number that has 1 divisor which is itself.
- All positive integers larger than 1 have at least two positive divisors.
- If a positive integer has only two positive divisors (itself and 1), then we call it a **prime** number.
- All positive integers larger than 1 and are not primes are called **composite** numbers.

From the previous definitions, we conclude the following:

- The number 1 is neither a prime nor a composite number.
- There is only one even prime number that is 2. And it is the smallest prime.
- The smallest composite number is 4.

All positive numbers N greater than 1 can be factorized into its prime factors. That is:

$$N = p_1^{\alpha_1}.\, p_2^{\alpha_2}.\, p_3^{\alpha_3} \ldots .\, p_k^{\alpha_k} \text{ Such that } p_1, p_2, p_3, \ldots, p_k$$

Are different primes. And $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$ are non-negative numbers. Then, the number of divisors of N is: $(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) \ldots (\alpha_k + 1)$.

Example 1: If $p, p^3 + 5$ Then $p^5 + 7$ is:

(a) Prime (b)composite (c) Prime or composite (d) Not A prime nor a composite

Solution: (b)

Notice that if p is odd then $p^3 + 5$ is an even number larger than 2 and that makes a composite number. Thus $p = 2$, which makes $p^5 + 7 = 2^5 + 7 = 39 = 13 \times 3$ which is a composite number.

Example 2: Given three prime numbers p, q, r which satisfy the condition

$$p + q = r, p < q$$

Find the value of p.

Solution: Notice that r cannot be even since the only even prime is 2 and it cannot be the sum of two even primes. Then, r is odd, which makes p, q one even and one odd number. However, we know that the only even prime is 2 . Moreover, $p < q$, which leads to $p = 2$.

Exercises:

1) Given x, y are prime numbers. Find the number of ordered solutions such that $x + y = 75$.

(a) 1 (b) 2 (c) 3 (d) 4

2) We call the two digit prime number $\underline{ab}$ a naughty number if $\underline{ba}$ is also prime. Find the number of naughty prime numbers.

3) Given the different prime numbers a, b, c that satisfy the equation $ab^b c + a = 2000$. Find all possible values of $a + b + c$.

4) Given that $p, q, p - q$ are primes, and $p + q$ is even. Find the value of:

$$(1 + \frac{1}{2})^p (1 - \frac{1}{3})^q$$

5) Challenge: If you knew that n is a positive integer such that $n + 3, n + 7$ are both primes. Find the remainder of n when divided by 3.

6) Challenge: if p is a prime that not less than 5, and $2p + 1$ is also a prime. Prove that $4p + 1$ is a composite.

7) We have the equation $56a = 65b$ with a, b positive integers. Prove that $a + b$ is a composite number.

8) We have $m > n$ two positive numbers that also satisfy $m^2 - n^2 - 2m - 2n = 19$.
Find the value of m, n.

9) Given three primes p, q, r that satisfy $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1661}{1986}$ find $p + q + r$.

10) Given m, n are two different primes. And p is the smallest integer that satisfies $p = m + n + mn$.
Find the value of $\frac{m^2+n^2}{p^2}$.

11) Given p, q are primes, and $p = m + n, q = mn$ such that m, n are positive integers. Find the value

$$\frac{p^p + q^q}{m^n + n^m}$$

12) Given p, q primes that satisfy the equation $5p + 7q = 129$. Find the values of $p + q$.

13) Challenge: Given $p, p + 2, p + 6, p + 8, p + 14$ are all prime numbers. Find the value of p.

14) Given p, q are two consecutive primes. The integer n satisfies that:

$$\{n - 1, 3n - 19, 38 - 5n, 7n - 45\} = \{p, 2p, q, 2q\}$$

Not necessarily in this order. Find the value of n.

15) Challenge: Prove that there exists an infinite number of values for n such that $n^2 + n + 41$ is a composite number.

Fourth Unit: Combinatorics

NSMO

Mawhiba

وزارة التعليم
Ministry of Education

Permutations

Permutation: is when we have a set of n distinct objects and we want to choose k of them, taking order into account and without repetition. The number of possible arrangements is given by:

$$^{n}P_{k} = n \times (n-1) \times (n-2) \times \dots \times (n-k+1)$$

and is read as "n permute k."

After learning about the concept of permutations, let's look at the following exercise:

Example:

In how many ways can the friends Ahmed, Mohammed, Younes, Hamza, and Hashem stand in a row for a group photo? In how many ways can 3 of them stand in a row? And in how many ways can at least 3 of them stand in a row?

Solution:

When arranging all five friends in a row: $^{5}P_{5} = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

When arranging only three of them: $^{5}P_{3} =$ **60**

When arranging at least three of them $= {}^{5}P_{3} + {}^{5}P_{4} + {}^{5}P_{5} = 60 + 120 + 120 =$ **300 ways**.

Note: We can write $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$ as $5!$, read as "five factorial."

Try calculating $4!$ and $3!$, what do you notice?

Note: For any positive integer n, We have $n! = (n-1)! \cdot n$

Factorial: The factorial of n is the product of all positive integers less than or equal to n, and it is denoted by the exclamation mark (!):

$$n! = n \times (n-1) \times (n-2) \times \dots \times 3 \times 2 \times 1$$

Notice that:

$$^{n}P_{n} = n \times (n-1) \times (n-2) \times \dots \times 3 \times 2 \times 1$$

represents the number of ways to arrange n objects in a row, taking order into account.

Exercises:

(1) A football stadium has fifteen gates. In how many ways can you enter through one gate and exit through another?

(2) In a conference hall, there are 7 lights. To keep the lighting suitable, at least one light must be on. In how many ways can this be done?

Note: Using the same reasoning as in the previous exercise, we can directly deduce from the multiplication principle that the number of subsets of a set containing n elements is: 2^n .

Since each element has two possibilities, it either belongs to the subset or does not. So:

Number of subsets: A set with n elements has 2^n subsets.

Exercises:

(3) Given the set $\{1, 2, 3, \dots, 8\}$:

(a) How many subsets does it have?

(b) How many subsets contain no odd numbers?

(c) How many subsets contain exactly one odd number?

(4) A square board is divided into identical small squares, with dimensions 2026×2026. We want to shade squares on the board so that exactly one square is shaded in each row and in each column. (Assume that rotation and reflection give different patterns.)

(a) In how many ways can this be done?

(b) If shading in the corner squares is not allowed, in how many ways can it be done?

(5) How many four-digit numbers are there that have exactly two identical digits, and whose thousands digit is less than 3?

(6) From a set containing 9 elements, how many non-empty subsets contain an even number of elements?

Properties of Permutations

Permutations with Repetition:

(7) Find the number of permutations of the letters of the word $PARALLEL$.

In the previous exercise, we noticed that some arrangements are counted repeatedly when using the ordinary permutation method, because swapping the first and second A makes no real difference, and similarly for the letter L.

Permutations with repetition:

If there are n objects where one item is repeated r_1 times, a second item r_2 times, and a third r_3 times, then the number of permutations is:

$$\frac{n!}{r_1! \times r_2! \times r_3!}$$

Now let's move to some exercises:

(7) How many ways are there to rearrange the letters of abcaadbddd?

(8) How many ways are there to rearrange the digits of 456733727?

(9) We have the digits 1, 2, 3, 1, 4, 5, 5:

(a) How many numbers can be formed by arranging all these digits?

(b) How many numbers can be formed by arranging all these digits, provided the number begins and ends with 5?

(10) How many words can be formed from the letters of the word SAUD?

(11) In how many different ways can the letters of SALMAN be arranged?

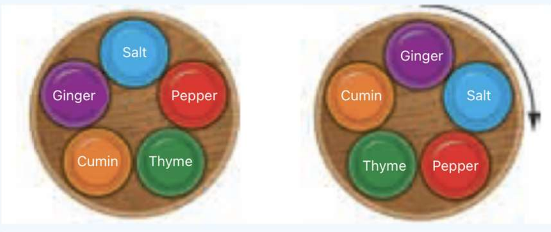
(12) How many different ways can the letters of the word ARABIA be arranged if the letters B must be between two A's (not necessarily next to them)?

(13) How many different words can be formed from the letters of ELEMENTARY if the three E′s appear together?

(14) How many ordered triples of positive integers (a, b, c) satisfy: $a \times b \times c = 231$

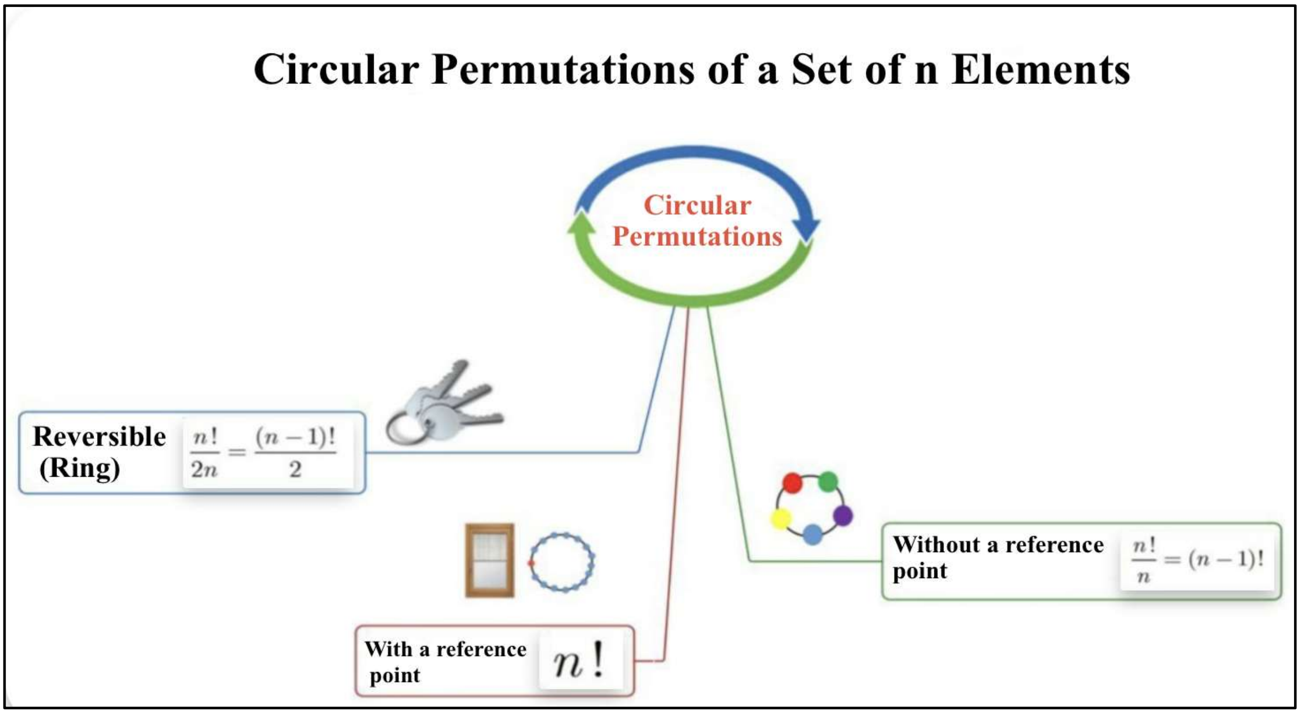
Circular Permutations

(18) In how many ways can we arrange five spice jars (salt, ginger, cumin, pepper, thyme) in a circle?



Circular permutation rule: The number of distinct arrangements of n objects placed around a circle is:

$$\frac{n!}{n} = (n-1)!$$



Exercises:

(16) In how many ways can we arrange 5 objects around a circle? 7 objects around a circle? n objects around a circle?

(17) In how many ways can eight people sit around a round table if one chair is red and the rest are black?

(18) In how many ways can we arrange:

(a) 36 people around a circle?

(b) 7 players in a circle if one of them is standing behind the referee?

(19) In how many ways can 5 students sit around a round table? And how many ways are there if one of the chairs is under the window?

(20) A circular necklace is made up of different types of beads. How many different necklaces can be formed from 13 beads:

(a) If flipping (reflection) is not allowed?

(b) If flipping is allowed?

(21) In how many ways can 5 engineers and 5 doctors be seated around a round table so that no two engineers sit next to each other?

(22) In how many ways can 7 engineers and 5 doctors be seated around a round table so that no two doctors sit next to each other? And how many ways if no two engineers are allowed to sit next to each other?

Solutions

Algebra Solutions

One-Variable Linear Equations:

Exercises:

(1)

Since the given equation contains complex fractions in both sides,

it is better to simplify each side separately first. From

$$1-\frac{x-\frac{1+3x}{5}}{3}=1-\frac{5x-(1+3x)}{15}=\frac{15-2x+1}{15}=\frac{16-2x}{15}\quad\frac{x}{2}-\frac{2x-\frac{10-6x}{7}}{2}$$
$$=\frac{x}{2}-\frac{14x-(10-6x)}{14}=\frac{10-13x}{14}$$

it follows that

$$\frac{16-2x}{15}=\frac{10-13x}{14}$$
$$\Longrightarrow 14(16-2x)=15(10-13x)$$
$$\Longrightarrow 224-28x=150-195x$$
$$\Longrightarrow x=-\frac{74}{167}$$

(2)

By moving 3 in the given equation to the left hand side, it follows that

$$\left(\frac{x-a-b}{c}-1\right)+\left(\frac{x-b-c}{a}-1\right)+\left(\frac{x-c-a}{b}-1\right)=0$$
$$\Longrightarrow \frac{x-a-b-c}{c}+\frac{x-a-b-c}{a}+\frac{x-a-b-c}{b}=0$$
$$\Longrightarrow (x-a-b-c)\left(\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right)=0$$
$$\because \frac{1}{c}+\frac{1}{a}+\frac{1}{b}>0, \therefore x-a-b-c=0$$
$$\Longrightarrow x=a+b+c$$

(3)

$$(x-3)^2+(x+1)^2+(4x-5)^2=0$$

We observe that each term is a non-negative quantity, and their sum equals **zero**. For this to occur, each term must be **zero** at the same time. However, this requires:

$$x=3, x=-1, x=\frac{5}{4}$$

simultaneously, which is impossible. Therefore, there is **no real solution (nor even a complex common solution)** to this equation.

(4)

Removing the denominator of the given equation yields

$$20(ax+b)-4(5x+2ab)=520ax+20b-20x-8ab=520(a-1)x=5-20b+8ab$$

$i)$ When $a\neq 1$:

$$x=\frac{5-20b+8ab}{20(a-1)}$$

$ii)$ When $a=1$ and $b=\frac{5}{12}$:

the equation becomes

$$0\cdot x=0$$

so, any real number is a solution for x.

(iii) When $a=1$ and $b\neq\frac{5}{12}$:

the equation becomes

$$0\cdot x=5-12b$$

so, no solution for x.

(5)

Change the given equation to the form $(2a+3b-12)x=5-3a$ we have

$$2a+3b-12=0 \qquad and \qquad 5-3a=0$$

Therefore

$$a=\frac{5}{3} \qquad , \qquad b=\frac{12-2a}{3}=\frac{26}{9}$$

(6)

From the given equation

$$2a(x+6) = 4x + 1$$

$$\Rightarrow (2a-4)x = 1 - 12a$$

Since it has no solution, this implies

$$2a - 4 = 0 \quad and \quad 1 - 12a \neq 0$$

$$\therefore a = 2$$

(7)

From

$$x = \frac{12}{k}$$

which is a positive integer, k is also positive integer and k is a divisor of 12. So

$$k = 1,2,3,4,6,12$$

The number of possible values of k is 6 .

(8)

$$\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2} = \frac{13}{12}$$

$$\Rightarrow \frac{3}{x+2} < \frac{13}{12} < \frac{3}{x}$$

$$\Rightarrow \frac{x+2}{3} > \frac{12}{13} > \frac{x}{3}$$

$$\Rightarrow x + 2 > \frac{36}{13} > x$$

$$\Rightarrow 13(x+2) > 36 > 13x$$

$$\Rightarrow x < 3$$

$$\therefore x = 1\,,2$$

By checking, $x = 1$ does not satisfy the original equation, and $x = 2$ satisfies the given equation. Thus, $x = 2$ the only positive value that satisfies the equation.

(9)

Substituting 4 into the given equation as x, it follows that

$$3a - 4 = \frac{4}{2} + 3 \Longrightarrow 3a = 9 \Longrightarrow a = 3$$

$$\therefore (-a)^2 - 2a = 9 - 6 = 3$$

(10)

$$\frac{n(x-n) - m(x-m)}{mn} = \frac{m}{n}$$

$$\Longrightarrow (n-m)x - n^2 + m^2 = m^2$$

$$\Longrightarrow (n-m)x = n^2$$

When $n \neq m$, we have

$$x = \frac{n^2}{n-m}$$

When $n = m$, no solution.

Multi-Variable Linear Equations:

Exercises:

(1)

$$\begin{cases} \frac{x-y}{5} - \frac{x+y}{4} = \frac{1}{2} \longrightarrow (1) \\ 2(x-y) - 3(x+y) + 1 = 0 \longrightarrow (2) \end{cases}$$

Simplifying the first equation, we have

$$\frac{x-y}{5} - \frac{x+y}{4} = \frac{1}{2} \Longrightarrow 4(x-y) - 5(x+y) = 10 \Longrightarrow x + 9y = -10 \longrightarrow (3)$$

Simplifying the second equation, we have

$$2(x-y) - 3(x+y) + 1 = 0 \Longrightarrow x + 5y = 1 \longrightarrow (4)$$

By $(3) - (4)$,

$$4y = -11 \Longrightarrow y = -\frac{11}{4}$$

From (4),

$$x = 1 - 5y = 1 + \frac{55}{4} = \frac{59}{4}$$

(2)

$$\begin{cases} 5.4x + 4.6y = 104 \longrightarrow (1) \\ 4.6x + 5.4y = 96 \longrightarrow (2) \end{cases}$$

Notice the feature of coefficients, by $(1) + (2)$, we obtain

$$10x + 10y = 200 \Longrightarrow x + y = 20 \longrightarrow (3)$$

By $(1) - (2)$, it follows that

$$0.8x - 0.8y = 8 \Longrightarrow x - y = 10 \longrightarrow (4)$$

By $(3) + (4)$, we obtain

$$2x = 30 \Longrightarrow x = 15$$

From (3),

$$x + y = 20 \Longrightarrow 15 + y = 20 \Longrightarrow y = 5$$

(3)

$$\begin{cases} x + 2(5x + y) = 16 \longrightarrow (1) \\ \quad 5x + y = 7 \longrightarrow (2) \end{cases}$$

By using 7 to substitute $5x + y$ in the first equation, we obtain

$$x + 2(5x + y) = 16 \Longrightarrow x + 2(7) = 16 \Longrightarrow x = 2$$

Then from the second equation,

$$5x + y = 7 \Longrightarrow 5(2) + y = 7 \Longrightarrow y = -3$$

(4)

$$\begin{cases} \frac{x}{2} = \frac{y}{3} = \frac{z}{5} \longrightarrow (1) \\ x + 3y + 6z = 15 \longrightarrow (2) \end{cases}$$

Let,

$$t = \frac{x}{2} = \frac{y}{3} = \frac{z}{5} \Longrightarrow x = 2t\,, y = 3t\,, z = 5t$$

Substituting into the second equation, we have

$$x + 3y + 6z = 15 \Longrightarrow 2t + 9t + 30t = 15 \Longrightarrow t = \frac{15}{41}$$

Thus

$$x = \frac{30}{41}\,, y = \frac{45}{41}\,, z = \frac{75}{41}$$

(5)

$$\begin{cases} x + 2y = 5 \\ y + 2z = 8 \\ z + 2u = 11 \\ u + 2x = 6 \end{cases}$$

From the given equations we have the cyclic substitutions

$$x = 5 - 2y\,, \qquad y = 8 - 2z\,, \qquad z = 11 - 2u\,, \qquad u = 6 - 2x$$

By substituting them sequentially, we have

$$\begin{aligned} x &= 5 - 2y = 5 - 2(8 - 2z) = -11 + 4z = -11 + 4(11 - 2u) = 33 - 8u = 33 - 8(6 - 2x) \\ &= -15 + 16x \end{aligned}$$

Therefore

$$x = -15 + 16x \Longrightarrow x = 1 \Longrightarrow u = 4\,, z = 3\,, y = 2$$

(6)

$$\begin{cases} 5x - y + 3z = a \longrightarrow (1) \\ 5y - z + 3x = b \longrightarrow (2) \\ 5z - x + 3y = c \longrightarrow (3) \end{cases}$$

By $2 \times (1) + (2) - (3)$, it follows that

$$14x = 2a + b - c \Longrightarrow x = \frac{2a + b - c}{14}$$

By $2 \times (2) + (3) - (1)$, it follows that

$$14y = 2b + c - a \Longrightarrow y = \frac{2b + c - a}{14}$$

Similarly, by $2 \times (3) + (1) - (2)$, we have

$$14z = 2c + a - b \Longrightarrow z = \frac{2c + a - b}{14}$$

(7)

By substituting the solution $(2\,,1)$ into equations, we obtain

$$\begin{cases} 2a + b = 7 \longrightarrow (1) \\ 2b + c = 5 \longrightarrow (2) \end{cases}$$

After eliminating b, we obtain

$$4a - c = 9$$

(8)

$$\begin{cases} 3x - y = 5 \\ 2x + y - z = 0 \\ 4ax + 5by - z = -22 \end{cases} \quad , \quad \begin{cases} ax - by + z = 8 \\ x + y + 5 = c \\ 2x + 3y = -4 \end{cases}$$

From the first equation of the first system, we get $y = 3x - 5$

By substituting it into the last equation of the second system, it follows that

$2x + 3(3x - 5) = -4$

So $x = 1\,, y = -2$

Then, the second equation of the first system yields $z = 0$

Thus, from the second equation of the second system, $c = 4$

By solving the system

$$4a - 10b = -22, \qquad a + 2b = 8$$

the solution for a and b is obtained:

$$a = 2, b = 3.$$

Therefore, $a = 2\,, b = 3\,, c = 4$

(9)

$a)$ When

$$\frac{k}{6} \neq \frac{-1}{3}$$

i.e. $k \neq -2$ the system has unique solution

$$x = 0\,, y = -\frac{1}{3}$$

$b)$ When

$$\frac{k}{6} = -\frac{1}{3}$$

The system has infinitely many solutions.

$c)$ Thus, it's impossible that the system has no solution.

(10)

$$\begin{cases} 2020(x-y) + 2021(y-z) + 2022(z-x) = 0 \\ \\ 2020^2(x-y) + 2021^2(y-z) + 2022^2(z-x) = 2021 \end{cases}$$

Let $u = x - y\,, v = y - z\,, w = z - x$. Then $u\,, v, w$ satisfy the following system of equations

$$\begin{cases} u + v + w = 0 \longrightarrow (1) \\ 2020u + 2021v + 2022w = 0 \longrightarrow (2) \\ 2020^2u + 2021^2v + 2022^2w = 2021 \longrightarrow (3) \end{cases}$$

By $2021 \times (1) - (2)$, we obtain

$$u - w = 0 \Longrightarrow u = w$$

From (1) again, we have $v = -2w$

By substituting it into (3), we have

$$(2020^2 - 2 \cdot 2021^2 + 2022^2)w = 2021$$

$$\Longrightarrow [(2022 + 2021) - (2021 + 2020)]w = 2021$$

$$\Longrightarrow 2w = 2021$$

$$\Longrightarrow z - y = -v = 2w = 2021$$

(11)

$$\begin{cases} x - y - z = 5 \longrightarrow (1) \\ y - z - x = 1 \longrightarrow (2) \\ z - x - y = -15 \longrightarrow (3) \end{cases}$$

By $(1) + (2) + (3)$,

$$x + y + z = 9 \longrightarrow (4)$$

By $(2) + (3)$, it follows that

$$-2x = -14 \Longrightarrow x = 7$$

By $(2) + (4)$ we obtain

$$2y = 10 \Longrightarrow y = 5$$

Similarly, by $(3) + (4)$ we obtain

$$2z = -6 \Longrightarrow z = -3$$

Thus, the solution is $x = 7$, $y = 5$, $z = -3$

(12)

$$\begin{cases} x - y + z = 1 \longrightarrow (1) \\ y - z + u = 2 \longrightarrow (2) \\ z - u + v = 3 \longrightarrow (3) \\ u - v + x = 4 \longrightarrow (4) \\ v - x + y = 5 \longrightarrow (5) \end{cases}$$

By adding the equations, we obtain

$$x + y + z + u + v = 15 \longrightarrow (6)$$

By $(1) + (2)$, $(2) + (3)$, $(3) + (4)$, $(4) + (5)$, $(5) + (1)$ respectively, we obtain

$$\begin{cases} x + u = 3 \\ y + v = 5 \\ z + x = 7 \\ u + y = 9 \\ v + z = 6 \end{cases}$$

By substituting into (6), we have

$$x = 0, y = 6, z = 7, u = 3, v = -1$$

Factoring:

Exercises:

(1)

$$x^2 - 8x + 7 = (x-1)(x-7)$$

(2)

$$x^2 - 6x - 7 = (x-7)(x+1)$$

(3)

$$x^2 - 25 = (x-5)(x+5)$$

(4)

$$2x^2 - 50 = 2(x^2 - 25) = 2(x-5)(x+5)$$

(5)

$$x^2 - 13x + 42 = (x-7)(x-6)$$

(6)

$$2x^2 + 5x + 2 = (2x+1)(x+2)$$

(7)

$$x^3 - 1000 = x^3 - 10^3 = (x-10)(x^2 + 10x + 100)$$

(8)

$$15x^3 + 7x^2 - 2x = x(15x^2 + 7x - 2) = x(5x-1)(3x+2)$$

(9)

$$5x^3 - 625 = 5(x^3 - 125) = 5(x^3 - 5^3) = 5(x-5)(x^2 + 5x + 25)$$

(10)

$$30x^4 + 5x^3 - 5x^2 = 5x^2(6x^2 + x - 1) = 5x^2(2x+1)(3x-1)$$

(11)

$$x^2 - 2x + 1 = (x-1)(x-1) = (x-1)^2$$

(12)

$$x^2 + 10x + 25 = (x+5)(x+5) = (x+5)^2$$

(13)

$$6x^3y - 13x^2y + 6xy = xy(6x^2 - 13x + 6) = xy(3x-2)(2x-3)$$

(14)

$$2x^2 + 10x + 12 = 2(x^2 + 5x + 6) = 2(x+3)(x+2)$$

(15)

$$12x^2y^2 - 15xy^2 - 63y^2 = 3y^2(4x^2 - 5x - 21) = 3y^2(x-3)(4x+7)$$

(16)

$$24x^3 + 10x^2y - 50xy^2 = 2x(12x^2 + 5xy - 25y^2) = 2x(4x - 5y)(3x + 5y)$$

(17)

$$9 - 4y^2 = 3^2 - (2y)^2 = (3 - 2y)(3 + 2y)$$

(18)

$$x^{12} - 1 = (x^6 - 1)(x^6 + 1) = (x^3 - 1)(x^3 + 1)((x^2)^3 + 1)$$

$$= (x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1)(x^2 + 1)(x^4 - x^2 + 1)$$

(19)

$$\frac{1}{4}a^4 - \frac{1}{9} = \left(\frac{a^2}{2}\right)^2 - \left(\frac{1}{3}\right)^2 = \left(\frac{a^2}{2} - \frac{1}{3}\right)\left(\frac{a^2}{2} + \frac{1}{3}\right) = \frac{1}{36}(3a^2 - 2)(3a^2 + 2)$$

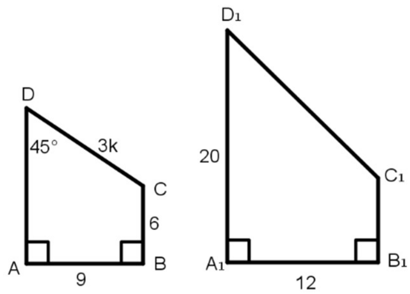
(20)

$$y^6 - 81 = (y^3)^2 - 9^2 = (y^3 - 9)(y^3 + 9)$$

Solutions of Similarity Exercises: $(1-8)$

Exercise	Polygons	Cannot be similar	Can be similar	Always similar
(1)	**Equilateral triangles**			√
(2)	**Isosceles triangles**		√	
(3)	**Squares**			√
(4)	**Rhombuses**		√	
(5)	**Right triangles**		√	
(6)	**Scalene Triangles**		√	
(7)	**Rectangles**		√	
(8)	**Right triangle and acute triangle**	√		

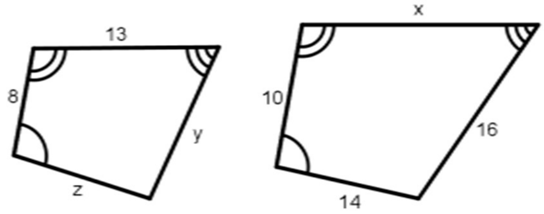
$(9-16)$



(9)	Similarity ratio of $ABCD$ ، $A_1B_1C_1D_1$.	$\frac{3}{4}$
(10)	Type of quadrilateral $A_1B_1C_1D_1$	Trapezoid
(11)	Measure of $m\angle D_1$	$45°$
(12)	Measure of $m\angle C_1$	$180-45=135°$
(13)	Measure of C_1B_1	$\frac{12}{9}=\frac{C_1B_1}{6} \Longrightarrow C_1B_1=8$
(14)	Measure of AD	$\frac{12}{9}=\frac{20}{AD} \Longrightarrow AD=15$
(15)	Measure of C_1D_1	$\frac{12}{9}=\frac{C_1D_1}{3k} \Longrightarrow C_1D_1=4k$
(16)	Ratio of the perimeters of the quadrilaterals	$\frac{3}{4}$

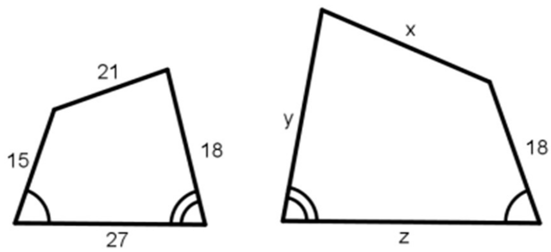
$(9-16)$

(17)



$$\frac{10}{8}=\frac{x}{13}\Longrightarrow x=\frac{65}{4}\,\frac{10}{8}=\frac{16}{y}\Longrightarrow y=\frac{64}{5}\,\frac{10}{8}=\frac{14}{z}\Longrightarrow z=\frac{56}{5}$$

(18)



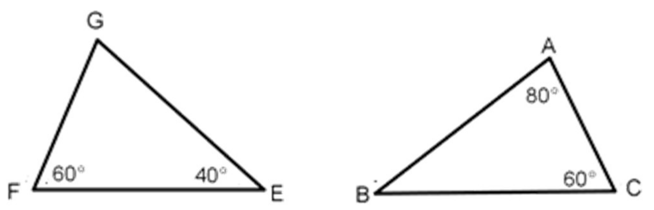
$$\frac{18}{15}=\frac{y}{18}\Longrightarrow y=\frac{108}{5}\,\frac{18}{15}=\frac{z}{27}\Longrightarrow y=\frac{162}{5}\,\frac{18}{15}=\frac{x}{21}\Longrightarrow x=\frac{126}{5}$$

Solutions of Triangle similarity exercises:

$(1-9)$

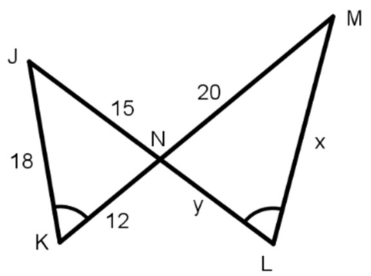
Exercise	Answer
1	Similar
2	Not Similar
3	Similar
4	Similar
5	Similar
6	Inconclusive
7	Similar
8	Inconclusive
9	Similar

(10)



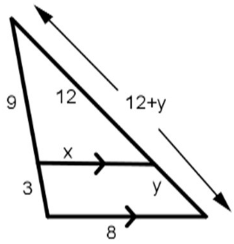
$$\angle ABC = 40°$$
$$\Delta ABC \sim \Delta GEF$$
$$\frac{AB}{GE} = \frac{BC}{EF} = \frac{AC}{GF}$$

(11)



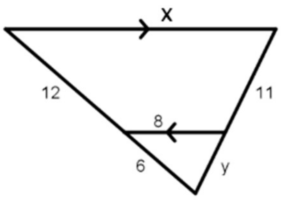
$$\Delta JKN \sim \Delta MLN$$
$$\frac{15}{20} = \frac{12}{y} \Longrightarrow y = 16 \frac{15}{20} = \frac{18}{x} \Longrightarrow x = 24$$

(12)



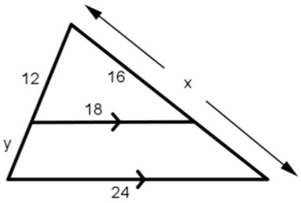
$$\frac{9}{12}=\frac{12}{12+y}\Longrightarrow y=3 \qquad \frac{9}{12}=\frac{x}{8}\Longrightarrow x=6$$

(13)



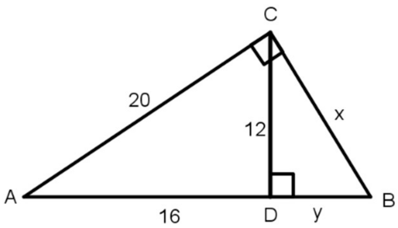
$$\frac{6}{18}=\frac{y}{11+y}\Longrightarrow y=\frac{11}{2} \qquad \frac{6}{18}=\frac{8}{x}\Longrightarrow x=24$$

(14)



$$\frac{18}{24}=\frac{12}{12+y}\Longrightarrow y=4 \qquad \frac{18}{24}=\frac{16}{x}\Longrightarrow x=\frac{64}{3}$$

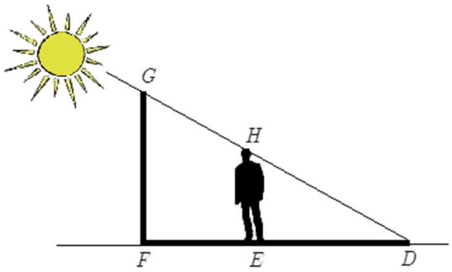
(15)



a) $\Delta ABC \sim \Delta ACD \sim \Delta CBD$

b) $\frac{BD}{CD}=\frac{CD}{AD}=\frac{CB}{AC}\Longrightarrow \frac{y}{12}=\frac{12}{16}=\frac{x}{20}\Longrightarrow x=15\ ,\ y=9$

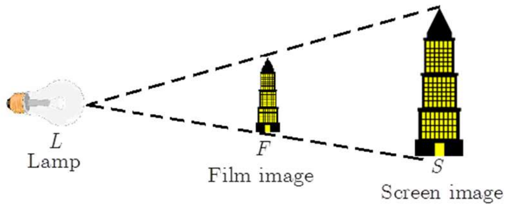
(16)



$$\Delta HDE \sim GDF$$

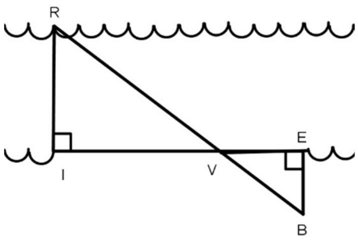
$$\frac{HE}{GF} = \frac{DE}{FD} \Longrightarrow \frac{2}{GF} = \frac{1.6}{4.4} \Longrightarrow GF = 5.5m$$

(17)



$$\frac{6\ cm}{24\ m} = \frac{x\ cm}{2.2\ m} \Longrightarrow x = 0.55\ cm$$

(18)



$$\Delta VEB \sim \Delta VIR$$

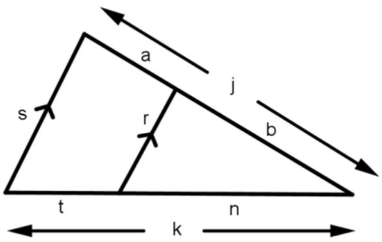
$$\Longrightarrow \frac{20}{63} = \frac{15}{IR} \Longrightarrow IR = 27.25\ m$$

(19 – 24)

Exercise	Similar Triangles	Theorem
(19)	$\Delta ABC \sim \Delta GNK$	***SAS***
(20)	$\Delta ABC \sim \Delta ARS$	***SAS***
(21)	$\Delta ABC \sim \Delta THJ$	***AA***
(22)	$\Delta ABC \sim \Delta MBL$	***SAS***
(23)	$\Delta ABC \sim \Delta XRN$	***SSS***
(24)	$\Delta ABC \sim \Delta AEF$	***AA***

Solutions of Proportional Lengths:

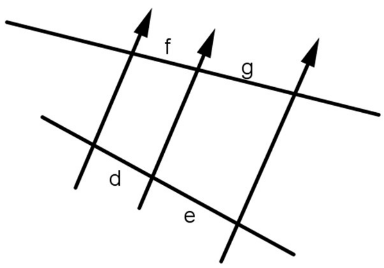
(1)



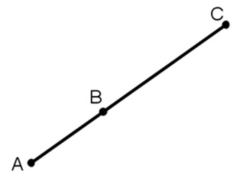
Part	Relation	Answer
(a)	$\frac{r}{s}=\frac{a}{b}$	X
(b)	$\frac{t}{k}=\frac{a}{j}$	√
(c)	$\frac{j}{a}=\frac{s}{r}$	√
(d)	$\frac{r}{s}=\frac{n}{k}$	√
(e)	$\frac{a}{b}=\frac{n}{t}$	X
(f)	$\frac{b}{j}=\frac{t}{k}$	X

(2)

Part	Relation	Answer
(a)	$\frac{d}{f}=\frac{g}{e}$	X
(b)	$\frac{f}{g}=\frac{e}{d}$	X
(c)	$\frac{g}{f}=\frac{e}{d}$	√
(d)	$\frac{d}{f}=\frac{e}{g}$	√

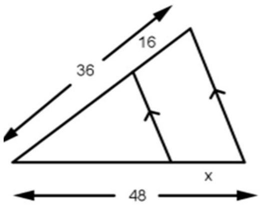


(3 – 6)



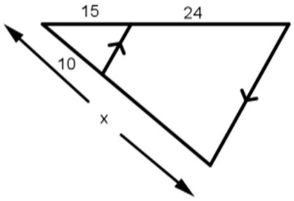
(6)	(5)	(4)	(3)	
37.5	21	15	6	AB
62.5	35	25	10	BC
100	56	40	16	AC

(7)



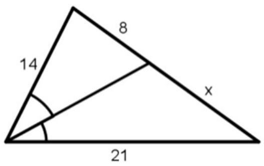
$$\frac{x}{48} = \frac{16}{36} \Longrightarrow x = \frac{64}{3}$$

(8)



$$\frac{10}{x} = \frac{15}{39} \Longrightarrow x = 26$$

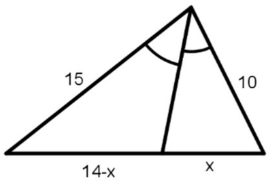
(9)



$$\frac{14}{21} = \frac{8}{x} \Longrightarrow x = 12$$

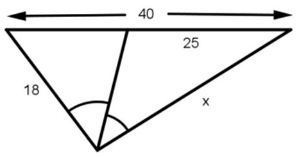
(10)

$$\frac{15}{10} = \frac{14-x}{x} \Longrightarrow x = \frac{28}{5}$$



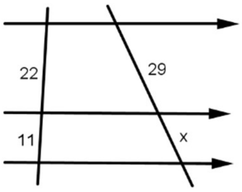
(11)

$$\frac{15}{25} = \frac{18}{x} \Longrightarrow x = 30$$



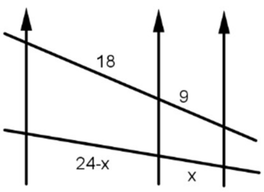
(12)

$$\frac{22}{11} = \frac{29}{x} \Longrightarrow x = 14.5$$



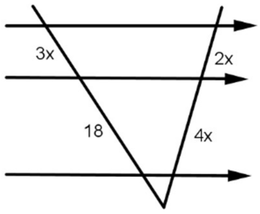
(13)

$$\frac{9}{18} = \frac{x}{24-x} \Longrightarrow x = 8$$



(14)

$$\frac{18}{3x}=\frac{4x}{2x}\Longrightarrow x=3$$



Solutions for Prime and Composite Numbers:

(1)

Answer is (b).

Notice that x, y sum up to an odd number (75). Thus, one of them is even and another is odd. But since they are both primes, it means one of them has to be 2 while the other is 73. And this gives the solutions x, y= (2,73), (73,2).

(2)

First, we can rule out the prime numbers with one digit equal to 2,4,5,6,8. After that, a simple search among the primes with digits in 1,3,7,9 Gives the numbers 11,13,17,31,37,71,73,79,97.

(3)

We have $a(b^b + c) = 2000 = 2^4 \cdot 5^3$. Thus, $a = 2\ or\ 5$.

Trying $a = 5$, we find that $b^b c = 399 = 3 \cdot 7 \cdot 19$ And this has no solutions.

Trying $a = 2$, we find that $b^b c = 3^3 \cdot 11$ Which has solutions only if $b = 3, c = 37$.

Thus, $a + b + c = 2 + 3 + 37 = 42$.

(4)

Since $p + q$ is even, then we have two cases:

Case 1: p, q are both even. Thus, $p = q = 2$ and this means that $p - q = 0$ which is not a prime. So this case has no solutions.

Case:2 p, q are both odd. This means that $p - q = 2$ and then we have $p = q + 2$. Plugging this into the equation we get that:

$$(1+\frac{1}{2})^p(1-\frac{1}{3})^q = (\frac{3}{2})^{q+2}.(\frac{2}{3})^q = (\frac{3}{2})^{q+2}.(\frac{3}{2})^{-q} = (\frac{3}{2})^2 = \frac{9}{4}$$

(5)

Notice that any positive integer can be written in one of the three forms $3m, 3m+1, 3m+2$ such that $m \geq 0$. Now, we have three cases:

Case 1: n is of the form $3m$. Then $n+3 = 3m+3 = 3(m+1)$ which is a multiple of 3.

Case 2: n is of the form $3m+2$ Then $n+7 = 3m+9 = 3(m+3)$ which is a multiple of 3.

Case3: n is of the form $3m+1$. Then $n+7 = 3m+8, n+3 = 3m+4$ which And these two numbers are possibly primes.

Indeed, if you plug $n = 4$ then $n+3 = 7, n+7 = 11$. Thus, the remainder of n when divided by 3 is 1.

(6)

When $p \geq 5$, then we can write p in the form $6k \pm 1$, such that $k \geq 1$ integer.

If $p = 6k+1$ then $2p+1 = 12k+3 = 3(4k+1)$ and this is a composite number.

On the other hand, if $p = 6k-1$ then $2p+1$ could be a prime number.

And $4p+1 = 24k-3 = 3(8k-1)$ is a composite number since it is a multiple of 3.

(7)

Since $\frac{a}{b} = \frac{65}{56}$. Then we can assume that $a = 65t, b = 56t$ such that t is a positive integer. Then we have $a+b = 65t+56t = 121t = 11^2 t$ which is a composite number.

(8)

We can rewrite the equation in the form: $(m-n)(m+n) - 2(m+n) = 19 \Rightarrow (m+n)(m-n-2) = 19$. It is clear that $(m+n) > (m-n-2)$. Thus, we have:

$$m+n = 19\,, m-n-2 = 1$$

Which gives solutions $m, n = 11{,}8$.

(9)

Notice that 1661, 1986 are relatively prime numbers (have no common factors. Moreover, the prime factorization of $1986 = 2 \times 3 \times 331$, we can notice that

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{331}=\frac{1661}{1986}$$

Thus, $p+q+r=2+3+331=336$.

(10)

Since p is the smallest possible integer such that $p=m+n+mn$, this means that m,n are the two smallest prime numbers which are 2,3. This means that $p=2+3+2\times 3=11$.

Therefore,

$$\frac{m^2+n^2}{p^2}=\frac{13}{121}$$

(11)

Since $q=mn$ is a prime number. This means that one of m,n is 1 while the other is q. Thus, $p=q+1$ are two consecutive prime integers. This can only be satisfied if $p=2, q=3$. This means that m,n are equal to 1 and 2.

(12)

Notice that if the sum of two integers is 129, this means that one of them has to be even. Therefore, one of p,q has to be 2 since they are prime numbers. Now, we have two cases:

Case 1: $p=2$ means that $q=17$ and therefore $p+q=19$.

Case 2: $q=2$ means that $p=23$ and therefore $p+q=25$.

This means that $p+q=\{19,25\}$.

(13)

It is clear that $p = 2{,}3$ does not satisfy the question statement. If $p = 5$ then 5,7,11,13,19 are all primes. Now, we have 5 cases:

Case 1: $p = 5k \neq 5$. Then p itself is not a prime since it will be divisible by 5.

Case 2: $p = 5k + 1$. Then $p + 14 = 5k + 15$ is not a prime since it will be divisible by 5.

Case 3: $p = 5k + 2$. Then $p + 8 = 5k + 10$ is not a prime since it will be divisible by 5.

Case 4: $p = 5k + 3$. Then $p + 2 = 5k + 5$ is not a prime since it will be divisible by 5.

Case 5: $p = 5k + 4$. Then $p + 6 = 5k + 10$ is not a prime since it will be divisible by 5.

Thus, the only solution is $p = 5$.

(14)

The only two consecutive prime integers are 2,3. Thus, $(n-1) + (3n-19) + (38-5n) + (7n-45) = 6n - 27$. Since p, q are 2,3 then $p + q + 2p + 2q = 15$. And from $6n - 27 = 15$ we get $n = 7$ and we just need to check that it is correct.

(15)

Notice that we can pick $n = 41k$. This makes $(41k)^2 + 41k + 41 = 41(41k^2 + k + 1)$. And this number is composite for any k. Thus, we are done.

Combintorics Solutions

Permutations:

(1)

$15 \times 14 = 210$

(2)

All subsets except the empty set (when all the lights are off): $2^7 - 1 = 127$

(3)

a) $2^8 = 256$

b) $2^4 = 16$

c) $4 \times 2^4 = 64$

(4)

(a) $2026!$

(b) We color the first row in 2024 ways (excluding the corner cells), and the last row in 2023 ways (excluding the corners and the cell chosen in the first row). For the remaining rows, there are $20240!$ ways. So Answer:

$$2024 \times 2023 \times 2024!$$

(5)

The thousands place can be 1 or 2, so there are 2 ways.

Case 1: If the thousands digit is the repeated digit, choose the repeated digit (2 ways), choose the second position for the repeated digit (3 ways), choose two different digits from the remaining digits ($8 \times 9 \div 2 = 36$ ways) and arrange them ($\times$ 2).
Total $= 2 \times 3 \times 36 \times 2 = 432$ ways.

Case 2: If the thousands digit is not the repeated digit, choose the thousands digit (2 ways), *choose the repeated digit from the remaining* 9 *digits (*9 *ways), choose the two positions for it among the remaining three places (*3 *ways), and choose one more different digit (*8 *ways).*
Total $= 2 \times 9 \times 3 \times 8 = 432$ *ways.*

Adding both cases: $432 + 432 = 864$ *numbers.*

Permutations with Repetition:

(6)

$$\frac{8!}{3! \times 2!} = 3360$$

(7)

$$\frac{10!}{4! \times 3! \times 2!} = 12600$$

(8)

$$\frac{9!}{3! \times 2!} = 30240$$

(9)

(a) $\frac{7!}{2! \times 2!} = 1260$

(b) $\frac{5!}{2!} = 60$

(10)

$$4! = 24$$

(11)

$$\frac{6!}{2!} = 360$$

(12)

The total number of arrangements without any restriction is $\frac{6!}{3!} = 120$

In all the words, the order of the letters A and B can be one of:

AAAB, AABA, ABAA, BAAA (temporarily ignoring R and I).

Thus, in half of the cases, the B is in the middle.

$$\frac{120}{2} = 60$$

(13)

The word ELEMENTARY has 10 letters, with repetitions as follows:

$$E \times 3, T \times 1, L \times 1, M \times 1, N \times 1, A \times 1, R \times 1, Y \times 1$$

If we require the three E's to be together, we treat them as a single block (EEE).

That means the word now effectively consists of 8 distinct elements, so the number of arrangements is:

$8! = 40320$

(14)

We factor the number 231 into its prime factors: $231 = 3 \times 7 \times 11$. Each prime factor can be distributed among a, b, and c independently. For each prime factor, there are 3 choices, it can go to a, b, or c. Thus, the total number of ordered triples is: $= 3^3 = 27$ ordered triples.

Circular Permutations:

(15)

$(5-1)! = 24$

(16)

5 objects: $(5-1)! = 24$ ways.

7 objects: $(7-1)! = 720$ ways.

For n objects: $(n-1)!$ ways.

(17)

$8! = 40{,}320$

(18)

(a) $(36 - 1)! = 35!$
(b) $7! = 5040$

(19)

a) $(5 - 1)! = 24$

b) $5! = 120$

(20)

a) If flipping the necklace is not allowed (i.e., the two directions are considered different): $(13-1)! = 12!$

b) If flipping is allowed (i.e., a necklace and its reversed version are considered identical), we divide the result by 2 $\frac{12!}{2}$,

(21)

We first arrange the doctors around the table. Since the seating is circular:

$$(5-1)! = 24 \text{ ways.}$$

There are now 5 empty seats between the doctors where the engineers can sit.

We arrange the engineers in those seats:

$$5! = 120 \text{ ways.}$$

Therefore, the total number of arrangements is: $24 \times 120 = 2880$ ways.

(22)

a) We first arrange the engineers around the table. Since the seating is circular: $(7-1)! = 720$ ways.

There are now 7 empty seats between the engineers where the doctors can sit.

We arrange the doctors in those seats

: ${}^{7}P_{5}$ ways.

Therefore, the total number of arrangements is: $720 \times {}^{7}P_{5}$ ways.

b) This is not possible because the number of empty seats between every two doctors is less than the number of engineers.

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